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OPTIMIZATION OF THE EXTRA-ATMOSPHERIC PART OF THE ASCENDING TRAJECTORY TO AN ORBIT

Christian Marchal

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OPTIMIZATION OF THE EXTRA-ATMOSPHERIC PART OF THE ASCENDING TRAJECTORY TO AN ORBIT
PART ONE - ELLIPTIC AND OPEN ORIENTATION CASE

Christian Marchal

ABSTRACT. For today's high thrust rockets (necessary for orbiting) the minimum expense of propellant is obtained by the use of minimum characteristic velocity solutions. When impulses are allowed, optimal ascents are always impulsional only. The "two-impulse ascending trajectory" with an immediate impulse and another impulse at the apogee of the final orbit, is generally the optimal solution. We use the usual notations (Chapter 1,2 and figure 1), subscripts 1 are referring to the fictitious launching orbit starting at the practical limit of the atmosphere (about 50 km altitude) subscripts 2 are referring to the target orbit.

INTRODUCTION

The ascent into orbit is an essential phase of any space mission /3*

The problem of optimal ascent into orbit includes two very different parts:

1. Optimization of crossing the atmosphere.
2. Optimization of the non-atmospheric phase of the putting into orbit.

The first problem, by far the most difficult of the two, has been taken under study by many writers (example: ref. [1]; it is clearly closely linked with aerodynamics and from this fact the minimum solutions vary a great deal depending on the vehicles contemplated. Nevertheless, some general laws and some limiting laws have been exhibited plainly.

The second problem, which is the one taken under study here, leads to identical solutions for all high-thrust vehicles. The characteristic optimized is, as always, the expenditure of propellant leading to minimum characteristic velocity solutions.

The study leads to two or three-impulse solutions, feasible in practice, with low loss, by means of thrust trajectories of short duration.

This problem has formed the subject of numerical studies [2] and also of some analytical studies in special cases.

In practice, of course, the two problems should be optimized at the same time: crossing of the atmosphere then extra-atmospheric flight path; this

*Numbers in the margin indicate pagination in the foreign text.

leads generally to "atmospheric exit" velocities slightly inclined to the horizontal. This is not, however, always the case, e.g., when only western launching azimuths are available. Also, out of concern for the general aspect, nothing particular will be assumed on the "atmospheric exit" velocity, considered as the initial velocity in the problem examined. /4

1.1 Description of the Theoretical Problem.

Upon exiting from the atmosphere (either at least from its aerodynamically important layers or from an altitude of about 40 to 70 km for ordinary rockets) a moving body is propelled by a determined velocity $\vec{V}_1$. now can it be caused to reach, with minimum expenditure of propellant, a final elliptical orbit with semimajor axis a_2 and eccentricity e_2 specified?

The orientation of the final orbit is considered to be open.

Of course, the fact that the earth rotates, as well as its atmosphere, does not become a factor in this part of the ascent into orbit. We shall also use only fixed axes (whose origin is the earth's center of gravity).

$\vec{V}_1$ is therefore the absolute exit velocity from the atmosphere.

We shall assume that velocity $\vec{V}_1$ is elliptical or parabolic. If, moreover, it were hyperbolic, the first thing to be done, in an optimum ascent, would be to restore it to a parabolic value either using retrorockets, or better, if it is possible, using an atmospheric braking in the minutes preceding the "exit from the atmosphere".

On the other hand, the gravitational potential of the earth will be assimilated to that of a uniform sphere with the same center and mass and we shall also disregard the effect of external celestial bodies.

In the case of vehicles for which maximum thrust corresponds to the use of maximum ejection velocity (chemical rockets and rockets with non-variable ejection velocity, etc...), the minimum expenditure for propellent still corresponds to the use of minimum characteristic velocity solutions, since the rocket is used according to the "all or nothing" method, i.e., according to a succession of maximum thrust trajectories separated by ballistic trajectories.

We shall therefore investigate the optimum ascents into orbit from the point of view of the minimum characteristic velocity. We shall first of all assume that we can, if necessary, produce instantaneous impulses, then we shall provide an order of magnitude for the loss (generally slight) in the case where the maximum thrust is limited.

We shall leave aside the rockets for which the maximum thrust does not correspond to the utilization of maximum ejection velocity (limited power nuclear electric rockets, etc...) for these rockets generally provide a weak thrust, unusable for an ascent into orbit.

1.2 Symbols

The first orbit, a launching orbit corresponding to velocity $\vec{V}_1$, is defined by its semimajor axis a_1 and its eccentricity e_1. Its apogee is outside the earth and its fictitious perigee is inside.

The final orbit aimed for is defined by its semimajor axis a_2 and its eccentricity e_2.

We shall utilize the customary symbols and reserve the non-indexed letters for the orbit depending on whether it is "real", "osculatory" or "instantaneous".

a = semimajor axis;

e = eccentricity;

$p = a\ (1-e^2)$ = parameter;

$b = a\ \sqrt{1-e^2}$ = semiminor axis;

ν = true abnormaltiy;

n = mean movement;

$\mu = n^2\ a^3$ = gravitational constant (= 398,580 km^3/sec^2 for Earth).

$\vec{r}$ = radius vector

$\vec{V}$ = velocity vector

$H = nab = |\Delta V|$ = kinetic energy

C = characteristic velocity = sum of all artifical velocity changes = $\int_0^t |\vec{\gamma}|\, dt + \Sigma\, |\vec{\Delta V}|$; ($\vec{\gamma}$ designating the accelerations of thrust and $\Delta\vec{V}$ the impulses).

P = distance from the perigee = $a(1-e)$;

A = distance from the apogee = $a(1+e)$;

Φ = angle of $\vec{V}$ and of the local horizontal plane $\left(\operatorname{tg}\Phi = \frac{e \sin \nu}{1 + e \cos \nu};\ -90^\circ \leqslant \Phi \leqslant +90^\circ\right)$.

Φ_1 = initial value of Φ = angle of "atmospheric exit".

R = radius of the atmosphere;
(= 6,370 km + 40 to 70 km in the case of Earth).

$L = \sqrt{\frac{2\mu}{R}}$ = escape velocity at the level of "atmospheric exit". (= 11.13 km/sec in the case of the Earth).

$K = \sqrt{\frac{\mu}{R}}$ = velocity of circular satellization at the level of "atmospheric exit".
(from Kreis = circle, in German)

$\left(K = \frac{L}{\sqrt{2}}\right.$ = 7.87/km/sec in the case of Earth).

$\vec{T}$ = velocity vector of ground surface at launching point

($\vec{T}$ is owing to the planet's rotation, T = 465 m/s x cosine of the local latitude in the case of the Earth).

1.3 Results.

(The proofs are set forth in Chapter 1.7).

In the impulse cases, the optimum ascents are of seven different types:

A. Case where atmospheric brakings are not used.

I. The Hohmann Transfer (Fig. 2). /5

This case is only found when $A_1 \geqslant A_2$.

A first impulse, tangential and accelerative, at A_1, makes transition to an intermediate ellipse with perigee P_2 ; a second impulse, tangential and braking, at P_2 makes transition to final orbit.

The characteristic velocity of this transfer is:

$$C_H = \sqrt{\frac{2\mu(A_1+P_2)}{A_1P_2}} - \sqrt{\frac{\mu(1-e_1)}{A_1}} - \sqrt{\frac{\mu(1+e_2)}{P_2}}.$$

II. The Ascent "Through Infinity" (Fig. 3).

A first immediate tangential impulse carries the value of the velocity of V_1 to L (parabolic velocity); at a great distance, at A, a negligible impulse allows redescent on the parabola or elongated ellipse AP_2 ; finally, at P_2 , a tangential braking makes the transition to the final orbit.

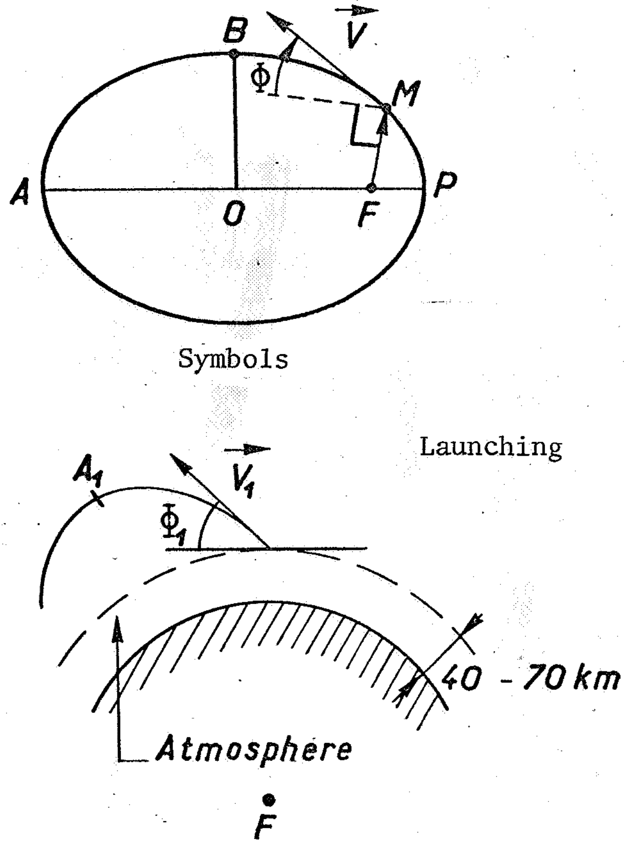

Fig. 1. OA = OP = a

OB = b; FP = P; FA = A; $\overrightarrow{FM} = \vec{r}$; $r = \dfrac{p}{1 + e\cos v}$

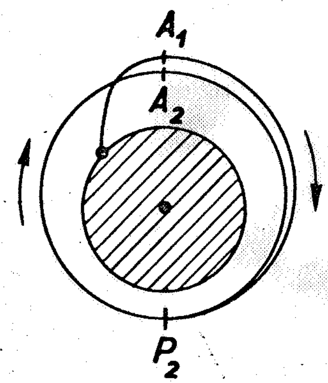

Fig. 2. Hohmann transfer.

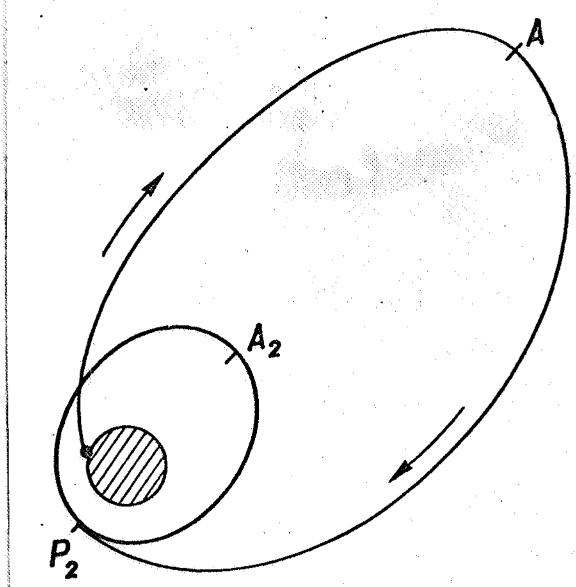

Fig. 3. Ascent "Through Infinity"

The characteristic velocity of this ascent is:

$$C_\infty = L - V_1 + \sqrt{\frac{2\mu}{P_2}} - \sqrt{\frac{\mu(1+e_2)}{P_2}}$$

Practical viewpoint:

It is clearly not possible to remove A to infinity; with fixed time of transfer or with fixed A distance the optimum is produced for all practical purposes when the two intermediate ellipses are coplanar and have A as apogee. The loss with respect to the optimum is then (for A major):

$$\delta C = \frac{RL}{2A}\left[\sqrt{\frac{P_2}{R}} - 1 - 2\cos\Phi_1\right]$$

(δC is positive only if $P_2 > R(1 + 2\cos\Phi_1)^2$ showing a first optimum condition for ascents "through infinity").

For $A = 60$ R (distance from the moon): $\frac{RL}{2A} = 93$ m/s, therefore δC is on the order of several tens of m/s.

Comment: it is possible to change the plane of final orbit for only a slightly higher cost by using at A an impulse not located in the initial plane.

III. The "Two-Impulse Ascent" (Fig. 4). /6

This case is only net if $A_1 < A_2$.

The first impulse is an immediate impulse. It transforms the velocity $\vec{V}_1 = \vec{OG}$ into $\vec{V} = \vec{OJ}$, point $\vec{J}$ being determined in the following manner:

1. The ellipse with center 0, with major horizontal axis MM' (with OM = $\sqrt{\frac{2\mu A_2}{R(A_2+R)}}$), with minor vertical axis NN' (with ON = $\sqrt{\frac{2\mu(A_2-R)}{A_2 R}}$) is the hodograph of velocities at 0 leading to an apogee at the distance from A_2. The foci of this ellipse are F_1 and F_2.

$$\left(OF_1 = OF_2 = \sqrt{\frac{2\mu R}{A_2(A_2+R)}}\right).$$

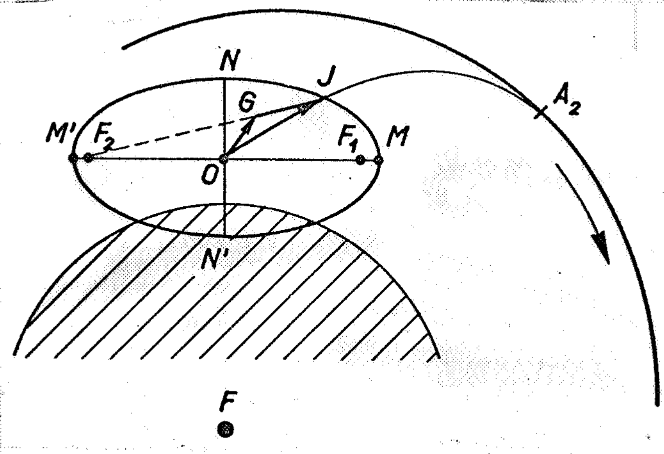

Fig. 4. The "Two-Impulse" Ascent.

2. Let F_2 be the focus farthest removed from G. Point J is on the above hodographic ellipse and on the straight line F_2G (G being between F_2 and J).

The velocity $\vec{V} = \vec{OJ}$ leads into an apogee A_2 where a second impulse (tangential and accelerating) makes transition to the final orbit.

The characteristic velocity of the "two-impulse ascent" is:

$$C_M = \sqrt{\frac{2\mu A_2}{R(A_2+R)}} + \sqrt{\frac{\mu(1-e_2)}{A_2}} - |F_2G|.$$

B. In the case where atmospheric brakings can be utilized at little cost, four new types of optimum solutions appear.

IV. The "Ascent Through the Atmosphere" (Fig. 5).

This case is only met if $A_1 > A_2$. At A_1, a tangential and accelerating impulse makes transition to the grazing orbit A_1P. At P an atmospheric braking leads to orbit PA_2. Finally, at A_2 a last tangential and accelerating impulse makes transition to the final orbit.

The characteristic velocity of this solution is:

$$C_{atm} = \sqrt{\frac{2\mu R}{A_1(A_1+R)}} - \sqrt{\frac{\mu(1-e_1)}{A_1}} + \sqrt{\frac{\mu(1-e_2)}{A_2}} - \sqrt{\frac{2\mu R}{A_2(A_2+R)}}.$$

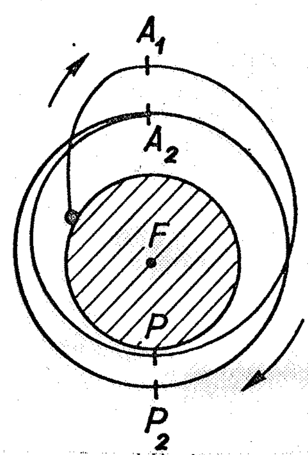

Fig. 5. Ascent "Through the Atmosphere".

V. The Ascent "Through Infinity, Then the Atmosphere" (Fig. 6).

It is enough to say that this solution starts as an ascent "through infinity" and ends as an ascent "through the atmosphere", with therefore and immediate tangential impulse with cost $(L-V_1)$, a very weak impulse at A, an atmospheric braking at P and a tangential and accelerating impulse at A_2.

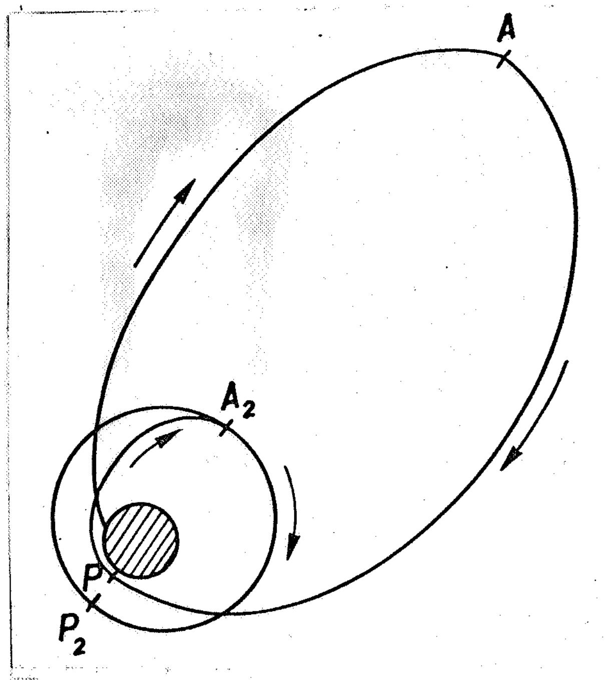

Fig. 6. Ascent "Through Infinity and then the Atmosphere".

The characteristic velocity of this solution is:

$$C_{\infty\,atm} = L - V_1 + \sqrt{\frac{\mu}{A_2}(1 - e_2)} - \sqrt{\frac{2\mu R}{A_2(A_2 + R)}}.$$

The loss owing to no movement apart from point A to infinity is:

$$\delta C = \frac{RL}{2A}(1 - 2\cos\Phi_1)$$

for A major (it is therefore necessary for Φ_1 to be greater than 60° for this transfer to be optimum.)

VI and VIII. /7

In order to be entirely complete it is advisable to mention two very unaccustomed cases of optimum utilization of atmospheric braking. In these cases, it is begun by reducing to zero the velocity with respect to the atmosphere using atmospheric braking. In other words, $\vec{V}_1$ is replaced by $\vec{T}$, then there being used either a "two-impulse ascent" (first case) or a "transfer through infinity" (second case).

Such solutions are always optimum if $V_1 < T$ (exceptional case). They can only be optimum if: $V_1 < L - K + T$; and if $V_1 \cos \Phi_1 < T$ (very rare case requiring a western launch azimuth and a low horizontal velocity of "atmospheric exit").

Comment: Of course in cases II, III, and V there are grounds for optimizing the altitude of the immediate impulse as a function of the atmospheric resistance to different velocities and different altitudes.

1.4 Discussion.

In a given case the various possible solutions must obviously be compared.

1.4.1 Case where Atmospheric Braking cannot be used.

Comparison should be made of the ascent "through infinity" and the "finite ascent" (i.e., the Hohmann transfer if $A_1 \geqslant A_2$ and the "two-impulse ascent" if $A_1 < A_2$).

Here are some simple rules:

Let us state:

$$p = \frac{2\,P_2}{1 + \dfrac{P_2}{\max(A_1;\,A_2)}};$$

if: $[\text{si } A_1 \leqslant A_2 : p = p_2 = a_2\,(1 - e_2^2)]$.

if: $P_2 \leqslant R\,[1 + 2\cos\Phi_1]^2$ or else if: $\operatorname{Log}\dfrac{p}{2\,R} \leqslant \dfrac{2+\sqrt{2}}{2}\cos\Phi_1$, the optimum ascent is the "finite ascent".

if: $P_2 \geqslant 9\,R\left[1 + \dfrac{3{,}9\,R}{\max[A_1;\,A_2]}\right]$ the optimum ascent is "through infinity".

1.4.2 Case where Atmospheric Braking can be used.

(We shall disregard the last two solutions VI and VII).

$$1^{o}\ P_2 < 4\,R\left(1 + \frac{R}{A_2}\right) : C_{\infty\,\mathrm{atm}} < C_\infty \text{ et } C_{\mathrm{atm}} < C_H.$$

It is necessary therefore to compare the "ascent ∞ atm" with the "ascent atm". (If $A_1 \geqslant A_2$) or with the "two-impulse ascent" (if $A_1 < A_2$). Remember that for $\Phi_1 \leqslant 60°$ the "ascent ∞ atm" is never optimum.

$$2^{o}\ P_2 > 4\,R\left(1 + \frac{R}{A_2}\right) : \quad C_{\infty\,\mathrm{atm}} > C_\infty.$$

In the case $A_1 < A_2$, it is therefore necessary to compare the "ascent through infinity" and the "two-impulse ascent" (cf. paragraph 1.4.1).

In the case $A_1 \geqslant A_2$ there are three ascent methods in competition: the ascent "through infinity", the Hohmann transfer (whence utilization of the rules of paragraph 1.4.1.) and the "ascent through the atmosphere" (this latter is never optimum if:

$$P_2 > 4\,R\left(1 + 2{,}8\,\frac{R}{A_2}\right)\Bigg].$$

1.4.3 Comments

I. From a practical viewpoint, the optimum ascent is very often a "two-impulse ascent." It is indeed enough that the four broad conditions following be satisfied:

$$1^{o})\ A_1 \leqslant A_2;\quad 2^{o})\ \Phi_1 \leqslant 60^{o};\quad 3^{o})\ V_1 \cos\Phi_1 \geqslant T;$$
$$4^{o})\ P_2 \leqslant R\,[1 + 2\cos\Phi_1]^2.$$

II. The "two-impulse ascent" leads to an injection at the apogee of the orbit and not at the perigee as is frequently done for the sake of technical convenience. Fortunately, the difference is generally small: the loss (in characteristic velocity), clearly zero if $e_2 = 0$, never exceeds:

$$\frac{3}{4} K \frac{(A_2 - R)^2}{(A_2 + R)^2}$$

or 15 m/s for $A_2 = R + 650$ km.

III. If the different methods of optimum ascent into orbit are plotted on a diagram, the following flight paths are obtained (Fig. 7).

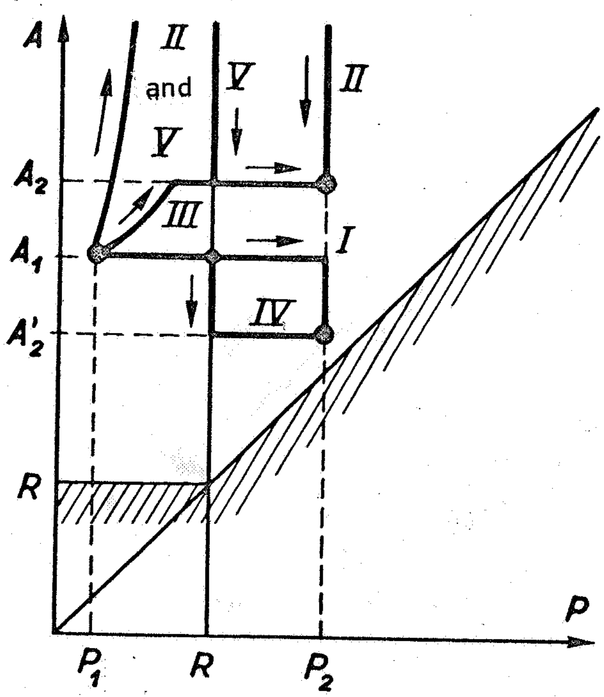

Fig. 7. Apogee-Perigee Diagram.

I. Hohmann transfer.
II. Ascent "through infinity".
III. "Two-impulse" ascent.
IV. Ascent "through the atmosphere."
V. Ascent "through infinity, then the atmosphere".

IV. We shall now study C, characteristic velocity, of $|\vec{V}_1'|$ putting into orbit as a function of the vector $\vec{V}_1$. /8

At $|\vec{V}_1|$ fixed, C becomes lower proportionally as Φ_1 drops. Therefore, at velocity $|\vec{V}_1'|$ with respect to the ground given, the optimum consists in a horizontal exit directed towards the east. This is approximately the solution used for customary satellite launchings.

The advantage of one launch site with respect to another (at equal altitudes) is therefore measured under these conditions by the difference of local velocities T of the ground (at the maximum 465 m/s between an equatorial site and a polar site). We shall see in the second part that equatorial sites are benefited much more if the orientation of the final orbit is no longer open.

Comment: If a site only has available western launch azimuths (for example, the Landes' base), it may be, at exit velocity $|\vec{V}'_1|$ with respect to the ground given, that the optimum is made up by an oblique or even vertical exit. These cases are plainly very unfavorable ones.

1.5 The Reverse Problem: Economical Descent from an Orbit.

We shall only consider the case where atmospheric braking can be used at little cost. In this case, it is required to reach at the least possible cost an orbit whose perigee is in the atmosphere. Two types of optimum solutions are easily obtained:

1) Case where $P_2 \leqslant 4R\left(1 + \frac{R}{A_2}\right)$.

A tangential braking at A_2 (Fig. 8), with cost: $C_D = \sqrt{\frac{\mu}{A_2}(1 - e_2)} - \sqrt{\frac{2\mu R}{A_2(A_2 + R)}}$

making transition to orbit A_2R.

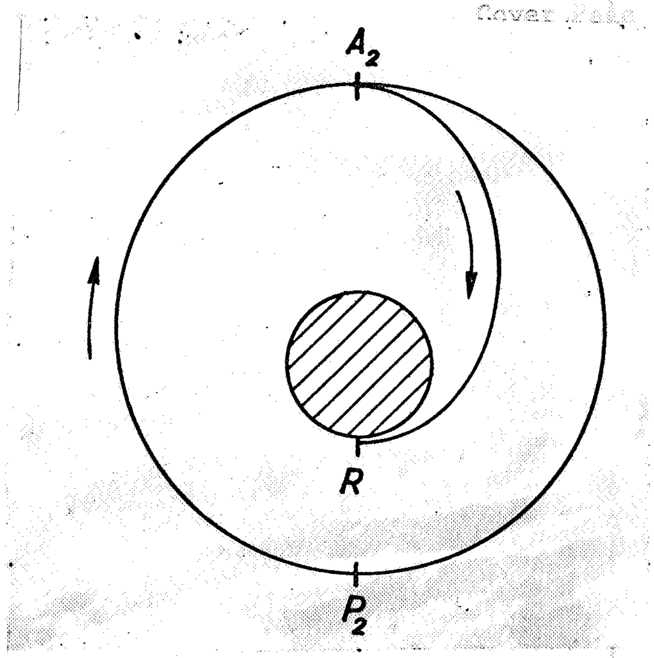

Fig. 8 Optimum Descent. First Case.

By adjusting atmospheric braking it is possible to land anywhere at all using the orbit A_2P_2 [3].

2) Case where $P_2 > 4R\left(1 + \frac{R}{A_2}\right)$.

A tangential acceleration at P_2 (Fig. 9), with cost:

$$C_{D\infty} = \sqrt{\frac{2\mu}{P_2}} - \sqrt{\frac{\mu}{P_2}(1 + e_2)}$$

leads to the very elongate ellipse P_2A. At A a negligible braking allows making transition to orbit AR.

The loss owing to not moving apart from point A at infinity is (for A major):

$$\delta C = \frac{L}{2A}\left(\sqrt{P_2 R} - 2R\right).$$

Comment: It is possible, for a hardly higher cost, to change plane at A and, hence, to land at any place whatsoever.

Example: descent from a synchronous satellite orbit ($A_2 = P_2 = 6.6$ R):
It follows that:

$$C_D = 1500 \text{ m/s} > C_{D\infty} = 1270 \text{ m/s}.$$

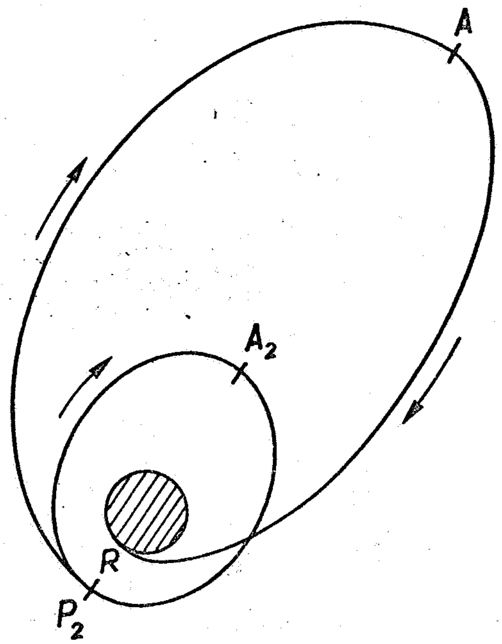

Fig. 9. Optimum Descent. Second Case.

The most troublesome descent corresponds to the case $A_2 = P_2 = (2 + \sqrt{2})$ R. It follows that:

$$C_D = C_{D\infty} = 1.490 \text{ m/s.}$$

By the way of comparison, the most troublesome of the optimum ascents into orbit (with an "exit angle" Φ_1 zero) is obtained for $A_2 = P_2 = 11.938$ R. It requires, from the ground: $C_2 = 12.13$ km/sec: if the launch base is polar.

$C'_2 = 12.13$ km/sec -0.465 km/sec $=$ 11.67 km/sec: if the launch base is equatorial.

1.6 Losses Owing to Limitation of Thrust. /9

1. Case of Non-Immediate Impulses.

If a non-immediate thrust is replaced by a thrust trajectory (optimally arranged) with duration t (from t_0 to $t_0 + t$), the relative loss $\frac{\delta C}{C}$ for this impulse is such that:

$$\frac{\delta C}{C} \leqslant \frac{2\pi^2}{T_1^2} \cdot \frac{C \int_{t_0}^{t_0+t} u^2 \gamma(u)\, du - \left[\int_{t_0}^{t_0+t} u \gamma(u)\, du\right]^2}{C^2}$$

T_1 designating the duration of rotation on a circular orbit at the altitude of the impulse and $\gamma(u)$ the value of acceleration at instant u,

(therefore $C = \int_{t_0}^{t_0+t} \gamma(u)\, du$).

If $\gamma(u)$ varies only a little: $\frac{\delta C}{C} \leqslant \frac{\pi^2}{6}\frac{t^2}{T_1^2}$.

(In the case of the Earth $\frac{\pi^2}{6}\frac{t^2}{T_1^2} < 10^{-3}$ if t = 2 min.)

2. Case of Immediate Impulses.

The loss is then higher. It is on the order of t instead of t^2.

A. Case of "Ascents through Infinity"

$$\frac{\delta C}{C} \sim \frac{(L - V_1)\sin \Phi_1}{2L} \cdot \frac{g}{\gamma}$$

(for γ major), with: g = acceleration of gravity (=9.8 m/s^2).

B. Case of the "Two-Impulse Ascent"

Let C_0 be the cost of the theoretical immediate impulse ($\vec{GJ}$ on Fig. 10) and φ the angle of this impulse with the horizontal plane; it follows that:

$$\frac{\delta C}{C} \sim \frac{\delta C}{C_0} \sim \frac{C_0 \sin^2 \varphi}{2\,[V_1 \sin \Phi_1 + C_0 \sin \varphi]} \cdot \frac{g}{\gamma} \cdot \frac{(A_2 - R)(A_2 + 2R)}{A_2 (A_2 + R)}$$

the relative loss is generally much less in this case than in the case of ascents "through infinity."

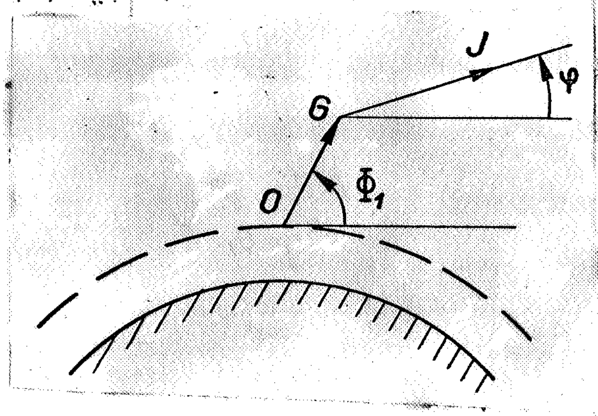

Fig. 10 "Two-Impulse" Ascent. Immediate theoretical impulse $\vec{GJ}$; $C_0 = |\vec{GJ}|$.

Comment: if $\sin \Phi_1 = 0$: $\sin \phi = 0$ and $\frac{\delta C}{C}$ is no longer of order $\frac{g}{\gamma}$ but of order $\frac{g^2}{\gamma^2}$ as in the case of non-immediate impulses.

1.7 Proofs

We shall only perform a proof in the case where atmospheric brakings cannot be used and where the rocket thrust is not limited (impulse case).

1.7.1 Examination of the Problem

Since only the shape of the orbits is taken into consideration, we can use a perigee-apogee diagram (Fig. 11).

One point of this corresponds to one single orbit (approximate orientations).

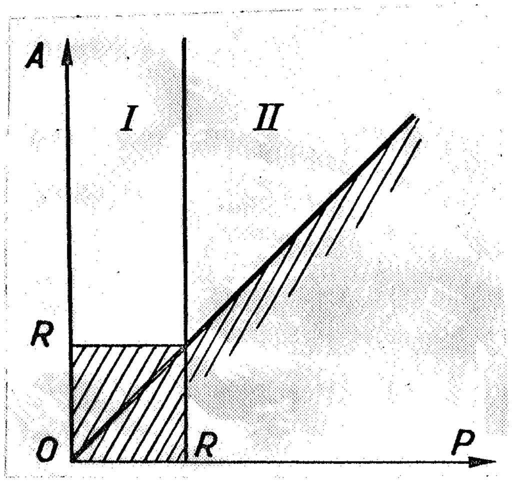

Fig. 11 Perigee-Apogee Diagram. Zone I and II.

Zone I (P<R<A) corresponds to secant orbits with the earth (or at least with the atmosphere).

We shall therefore look for the optimum path between a point of zone I and a point of zone II.

Since the orbits in zone I are interrupted by the atmosphere, the parameter ν (true anomaly) cannot be considered as an absolutely free control parameter. If, indeed, the optimum position of application of the thrust has been exceeded, it is not possible to return there. We shall assume, however, that this can be done and we shall certainly see if the result obtained is feasible.

It is then possible to prove successively:

1.7.2 Theorem I

An optimum path cannot descend in zone I.

Indeed, since the goal is [illegible] things in zone II:

1. Either, starting [illegible] path returns to the initial level at K (Fig. 12); the rectilinear flight path JK is in this case less costly (Hohmann transfer by an impulse at the apogee, therefore feasible).

2. Or, the path ends at M on the straight line P = R; the flight path JLM (Hohmann transfer feasible) is in this case less costly.

Theorem I is therefore quite exact.

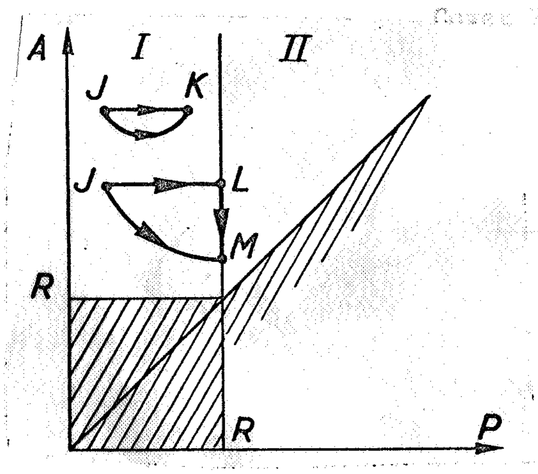

Fig. 12 Theorem I

1.7.3 Theorem II /10

The rocket can only be utilized the highest or lowest possible.

Let us now investigate the range of maneuverability.

The parameters of state can be a, H and C (semimajor axis, kinetic energy and characteristic velocity) since only the shape of the orbit becomes a factor.

The component of the thrust normal to the orbiting plane only modifies the orientation. The optimum thrusts are therefore in the plane of the launch orbit.

Let ϕ be the angle of thrust with the horizontal (we shall assume ϕ positive upwards) and it follows that:

$$\frac{dH}{dC} = r \cos \varphi$$

$$\frac{da}{dC} = \frac{2a^2}{H} [\cos \varphi + e \cos (\varphi - v)].$$

Pontryagin's optimum condition [4] is stated: maximum with respect to ϕ and ν $\mathcal{H} = p_H \frac{dH}{dC} + p_a \frac{da}{dC}$:

It is true that:

$$\mathcal{H} = \left[r p_H + \frac{2a^2}{H} p_a (1 + e \cos \nu)\right] \cos \varphi + \frac{2a^2 e}{H} p_a \sin \nu \sin \varphi$$

it therefore follows that:

$$1^{o})\quad \operatorname{tg} \varphi = \frac{\frac{2a^2 e}{H} p_a \sin \nu}{r p_H + \frac{2a^2}{H} p_a (1 + e \cos \nu)}$$

$$2^{o})\quad \Sigma = \left[\frac{2a^2 e}{H} p_a \sin \nu\right]^2 + \left[r p_H + \frac{2a^2}{H} p_a (1 + e \cos \nu)\right]^2 :$$

maximum with respect to ν.

$$\Sigma = p_H^2 r^2 + \frac{8}{r}\frac{a^4 p_a^2}{\mu} + \frac{4 p_a}{n^2}[n b p_H - p_a].$$

The last term is independent of ν and the maximum of $p_H^2 r^2 + \frac{8}{r}\frac{a^4 p_a^2}{\mu}$'s is produced either for r maximum (at the apogee) or for r minimum (r = R in zone I and r = P in zone II).

Theorem II is therefore quite correct.

It may be noted, owing to the expression above of the optimum value of tg ϕ, if the thrust takes place at the perigee or apogee ($\sin \nu = 0$), that tg $\phi = 0$ (horizontal thrust) is produced.

1.7.4 Theorem III

The switchings in zone I always take place in the direction $r = R \to r = A$.

For a switching in zone I, Σ has the same value for r = R and r = A therefore:

$$p_H^2 R^2 + \frac{8}{R}\frac{a^4 p_a^2}{\mu} = p_H^2 A^2 + \frac{8}{A}\frac{a^4 p_a^2}{\mu}$$

or

$$p_H^2 (A + R) = p_a^2 \frac{8 a^4}{\mu A R}$$

in which the limiting values of ϕ for r = R (switching values):

$$\operatorname{tg} \varphi_1 = \frac{e \sin \nu}{1 + e \cos \nu - \sqrt{\frac{2R(1-e)}{A+R}}},$$

and:

$$\operatorname{tg}\varphi_2 = \frac{e \sin \varphi}{1 + e\cos\varphi + \sqrt{\dfrac{2\,R\,(1-e)}{A+R}}}.$$

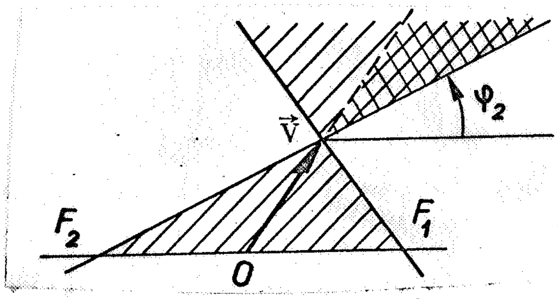

Fig. 13 Useful Angle at Level r = R.

The simple geometric construction above (Fig. 13) with the useful angle (cross-hatched) was derived with:

$$OF_1 = OF_2 = \sqrt{\frac{2\,\mu R}{A\,(A+R)}}$$

(in reality, only the double cross-hatched portion, under the tangent, is actually used).

As in zone I, A cannot decrease (theorem I), OF_1 and OF_2 cannot increase. A direction used as direction of thrust cannot enter into the useful angle, it can only exit from it. Therefore, the communications in zone I can only take place in the direction r = R → r = A. The theorem III is therefore quite correct.

In this way it is incidentally confirmed that the hypothesis: ν absolutely free control parameter does not lead to an impossibility.

1.7.5 Optimum Flight Paths

Taking into account optimum flight paths in zone II (Hohmann transfers and transfers "through infinity"), the optimum flight path between a point of zone I and a point of zone II can only have one of the three shapes below (Fig. 14).

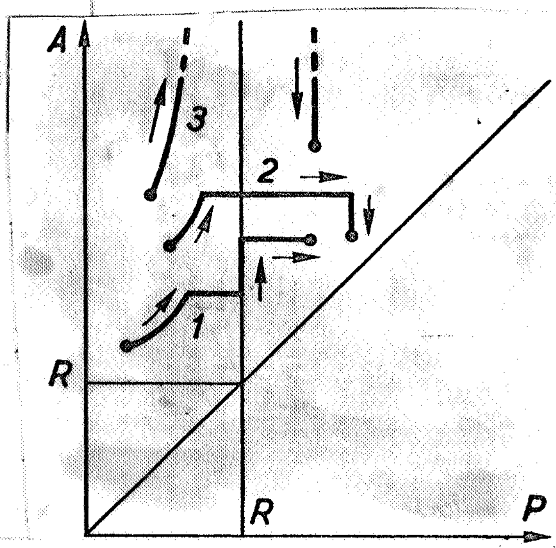

Fig. 14 Optimal Paths

Shape 3 corresponds to the "ascent through infinity" described in Chapter 1.3 from the results (the first impulse is a tangential immediate impulse carrying the velocity from V_1 to L (parabolic velocity).

1.7.5.1 First of all let us optimize the part of the transfer located in zone I. /11

A first impulse at level r = R takes the apogee to value A. A second impulse (tangential at the apogee) takes the perigee to value R (Fig. 15).

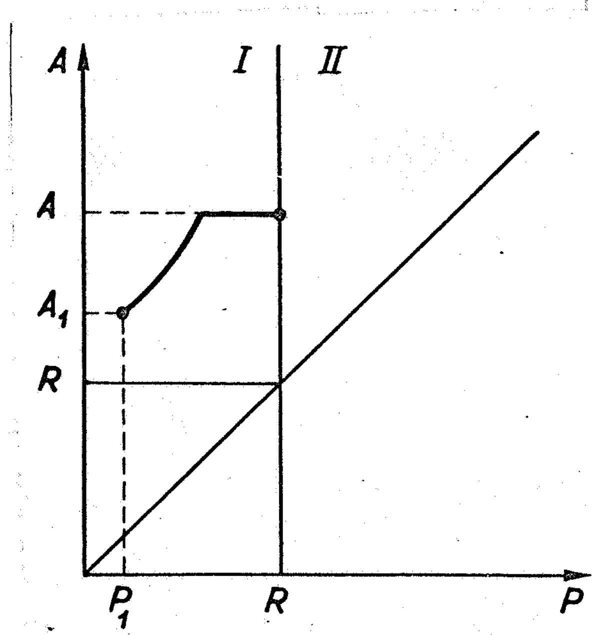

Fig. 15 Optimization in Zone I.

The hodograph of the velocities at 0 giving an apogee at distance A is an ellipse with center 0 (Fig. 16), major horizontal axis

$$MM' \left(OM = \sqrt{\frac{2\mu A}{R(A+R)}}\right),$$

minor vertical axis

$$NN' \left(ON = \sqrt{\frac{2\mu(A-R)}{AR}}\right),$$

foci

$$F_1 \text{and} F_2 \left(OF_1 = OF_2 = \sqrt{\frac{2\mu R}{A(A+R)}}\right);$$

The first impulse is therefore a vector GJ, J being a point of the ellipse MN M' N' and the cost of the portion of the transfer in zone I is:

$$C_I = |\vec{GJ}| + \sqrt{\frac{2\mu R}{A(A+R)}} - \frac{R}{A}|\vec{V}_a|.$$

But [illegible] is precisely the eccentricity of the ellipse MNM' N'; it follows that

$$\frac{R}{A}|\vec{V}_a| = ||OM| - |F_1J||.$$

The optimum position of J is therefore on the straight line F_2G (F_2 being the most removed focus of G) from the side of G. It is accordingly true that:

$$C_I = GJ + F_1J - OM + \sqrt{\frac{2\mu R}{A(A+R)}} = F_2M - F_2G.$$

There is recovered in this way a construction similar to the one shown in 1.3 for the "two-impulse ascent", (a construction satisfying the conditions of useful angle and switching of paragraph 1.7.4).

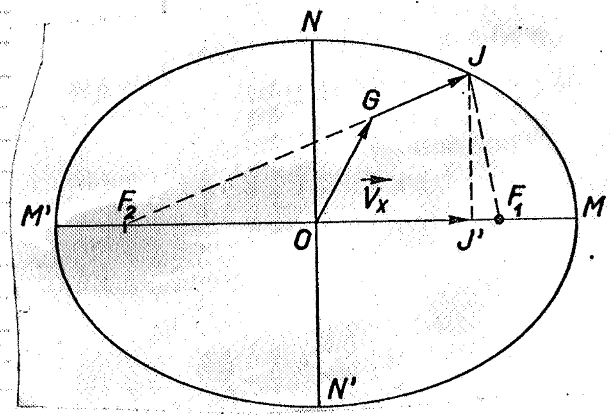

Fig. 16. Optimization in Zone I.

$\vec{OG} = \vec{V}_1$; $\vec{OJ'} = \vec{V}x$.

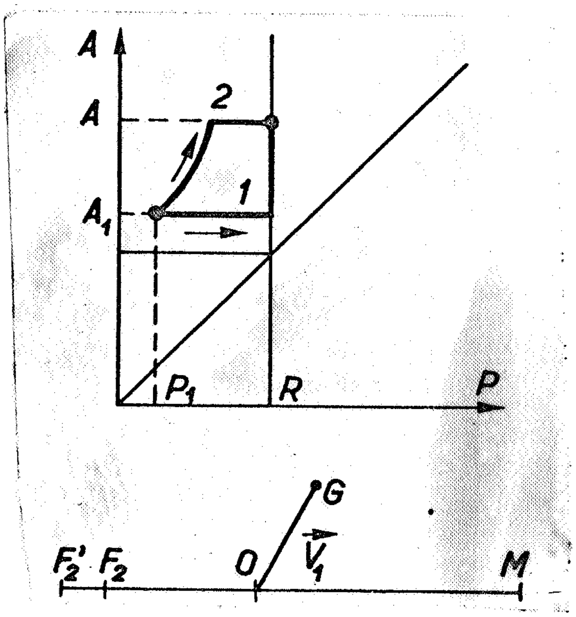

Fig. 17. Comparison Between Paths 1 and 2.

1.7.5.2 Comparison of paths 1) and 2) below (Fig. 17) always leads to the removal of path 1). /12

Indeed:

$$C_{1)} = F_2'M - F_2'G > C_{2)} = F_2M - F_2G$$

with

$$OF_2' = \sqrt{\frac{2\,\mu R}{A_1(A_1+R)}} > OF_2 = \sqrt{\frac{2\,\mu R}{A\,(A+R)}}.$$

1.7.5.3 The optimum flight paths between a point of zone I and a point of zone II therefore have the shape 2) or 3) (Fig. 14) and the only question remaining for solution is to determine the optimum ordinate A of the intermediate segment (Fig. 18).

1.7.6 Theorem IV.

The optimum value of A is either A maximum or A minimum, i.e., either A infinity or A = max (A_1, A_2).

The proof of this last theorem is quite complicated. We shall show only the following elements:

If ϕ_A is the angle determining the direction of the first impulse:

$$\operatorname{tg} \varphi_A = \frac{e_1 \sin v_1}{1 + e_1 \cos v_1 + \sqrt{\dfrac{2\,R\,(1-e_1)}{(A+R)}\,\dfrac{A_1}{A}}} = \frac{\sqrt{(A_1-R)\,(R-P_1)}}{b_1 + \sqrt{\dfrac{2\,R^3}{(a_1\,A)\,(A+R)}}}$$

it follows that (C_2 designating the total cost of the transfer):

$$\operatorname{sgn} \frac{dC_2}{dA} = \operatorname{sgn} \left[\frac{R}{P_2} - \frac{(A+R)^3}{(A+P_2)[A+(2A+R)\cos\varphi_A]^2} \right]$$

let us state

$$S = \frac{R}{P_2} - \frac{(A+R)^3}{(A+P_2)\,[A+(2A+R)\cos\varphi_A]^2}.$$

The proof consists in verifying that S<0 leads to $\frac{dS}{dA} < 0$ and therefore that S is not cancelled more than one and in this case in a decreasing direction. The same is therefore true of $\frac{dC_2}{dA}$ and the minimal value of C_2 is obtained for one of the extreme values of A consistent with theorem IV.

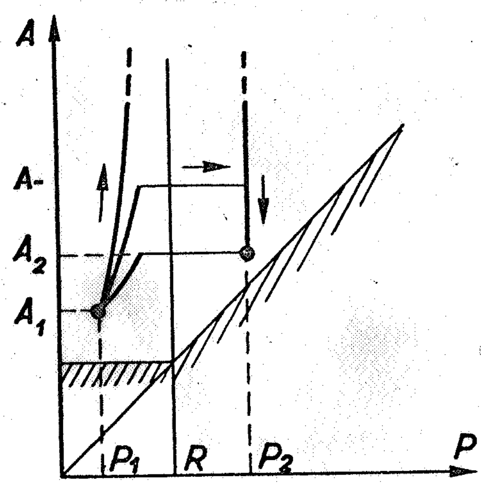

FIG. 18 Final Optimization.

In the hypotheses under which we operate here (capability of performing impulses, incapability of using atmospheric brakings), there is therefore in each case two competing solutions for comparison, always involving impulses: the ascent "through infinity" on one hand and the "Hohmann transfer" (if $A_1 \geqslant A_2$), or else the "two-impulse ascent" (if $A_1 < A_2$) on the other hand, just as it is explained and discussed in Chapter 1.3 and 1.4.

CONCLUSION

The optimization of the extra-atmospheric phase of the ascent into orbit (in the case where the final orbit is elliptical and with open orientation) leads to two or three-impulse solutions which can be carried out practically with a slight loss by means of thrust trajectories of short duration.

The optimum solution is almost always of the "two-impulse ascent" type with one impulse starting from the "exit from the atmosphere" and the other (injection impulse) tangential to the apogee of the target orbit.

In Part Two we shall study the case in which the target orbit plane is determined and not open. The equatorial launch sites in question have many more advantages with respect to the others than in the present case.

In Part Three we shall study the case in which the final orbit is hyperbolic.

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OPTIMIZATION OF THE EXTRA-ATMOSPHERIC PART OF THE ASCENDING TRAJECTORY TO AN ORBIT

PART TWO - CASE IN WHICH THE TARGET ORBIT IS ELLIPTIC AND WELL DETERMINED (FIXED-ORIENTATION CASE). COMPARISON BETWEEN THE LAUNCHING BASES

Christian Marchal

ABSTRACT. Optimization of the extra-atmospheric phase of ascent to orbit can be achieved by adopting certain simple assumptions regarding cost of travel through the atmosphere and the height of the dense atmosphere. Choice of the launch site depends on local latitude and available launch sectors. For inclinations of 5° to 117° obtainable at the best launching station the optimum ascending trajectory is always of the direct-climb type. For other less favorable inclinations this trajectory may be of the three-nodal climb, four-nodal climb, and other types. For the latter inclinations substantial saving can be effected by use of atmospheric braking and fast-climb trajectories with no intermediate parking orbit.

INTRODUCTION

The ascent into orbit is an essential phase of any space mission.

In many cases the orientation of the target orbit is not open either, for example, when it is desired to place a telecommunications satellite on a "geostationary" orbit, hence in the equatorial plane, or else a reconnaissance satellite on polar orbit, etc. The study made in Part One is then no longer adequate [1].

The general study of optimization of ascent into orbit is a very complicated problem, on one hand because of the crossing of the atmosphere and, on the other hand, because of the great number of parameters defining an orbit. It is nevertheless simplified if it is granted that:

1. The optimization of the crossing of the atmosphere leads to a consumption which is a function of the magnitude alone (and not of the orientation) of the velocity, with respect to the ground, of "atmospheric exit" (approximately 40 to 70 km of altitude in the case of the Earth).

2. The equatorial rotation velocity is small compared with the satellization velocity (this not being the case for the large planets). /4

The problem stated is then much simpler and, if the launch base is "all azimuths", it can be solved in a very great number of cases. The "all azimuths" restriction is in general only slightly restrictive for bases having available broad launching sectors either in the northeast quadrant or in the southeast quadrant.

The problem, practically solved if the inclination of the orbiting plane is greater than the latitude of the launch location, still remains to a great extent indeterminate in the contrary case (much worse case).

II.1. Statement of the Problem

How, beginning from a given launching base, is it possible to reach a fixed elliptical orbit with the minimum propellant consumption of a given rocket?

We shall assume:

A) That the planet from which the launching takes place is spherical and either devoid of atmosphere or supplied with an atmosphere of which the aerodynamically important part is not thick (40 to 70 km in the case of the Earth).

B) That the launching base is "all azimuths", (we shall make a rapid inspection of what may be seen in the reverse hypothesis).

C) That the cost of the optimized crossing of the atmosphere is independent of the orientation of the velocity, with respect to the ground, of "atmospheric exit" (consequently we shall not take it into account).

D) That the various perturbative effects (owing to the sun, equatorial bulge, etc.) are negligible.

The optimization leads to the utilization of ascents of characteristic minimum velocity for all high-thrust rockets (chemical rockets, nuclear rocket, etc...) whose use is required for an ascent into orbit. We shall therefore investigate these ascents under the assumption that the thrust power is not limited (impulse case). The loss owing to this limitation is generally of the second order with respect to the duration of the thrust trajectories (relative loss less than 10^{-3} in the case of the Earth for trajectories of 2 minutes duration). It can be of the first order for the first impulse if a transition is not made through an intermediate waiting orbit [1]

Note that the characteristic velocity is the arithmetic sum of all the artificial changes in velocity.

Optimization of the ascent sometimes depends on the possibility or im-

possibility of accomplishing atmospheric brak

The cases of finite lift-drag ratio are clearly located between cases II and III.

ing at low cost. Hence we shall

consider the following three cases:

I. Atmospheric braking cannot be accomplished.

II. Atmospheric braking can be accomplished, but the lift-drag ratio of the rocket considered is zero.

III. Atmospheric braking can be accomplished and the lift drag ratio of the rocket considered is infinite.

The last-named case, in which the rocket can proceed throughout the atmosphere without loss of velocity, renders all the stations equivalent to the best one, the equatorial station (for which a non-zero lift-drag ratio is not useful).

The cases of finite lift-drag ratio are clearly located between cases II and III.

On the other hand, it is possible to wish to make a transition through an intermediate parking orbit, for example, for matters of precision. We shall therefore also consider this case and we shall still assume that this parking orbit is located lower than the perigee of the target orbit (if not the consumption for ascent into orbit would be much higher).

11.2 Notation

We shall use the equatorial plane of the planet studied as reference plane and we shall employ the customary symbols.

The indices 1, 2, 3, 4...are related to the successive intermediate orbits and to their components, index B relating to the final orbit ("target" orbit) and to its components.

O = orbit; a = semimajor axis; e = eccentricity; $P = a(1-e)$ = distance from the perigee to the center of the planet; $A = a(1+e)$ = distance from the apogee to the center of the planet; i = inclination of the orbiting plane to the equatorial plane; ($0° \leqslant i \leqslant 180°$, the planes are oriented in the direction of rotations); Ω = right ascension of the ascending equatorial node;

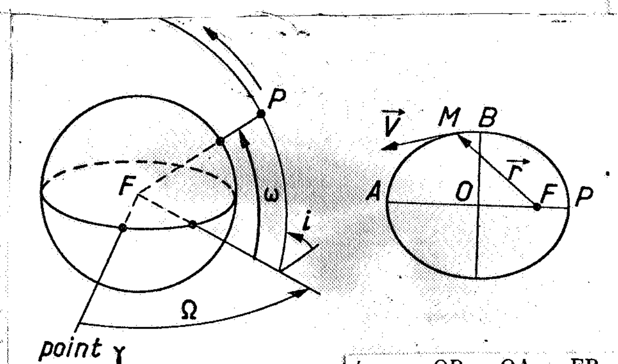

Fig. 1 Symbols $\left\{ \begin{array}{l} a = OP = OA = FB \\ b = OB \\ P = FP \\ A = FA \end{array} \right.$

ω = argument of the perigee (= angle between the ascending node and the perigee, in the direction of movement);

φ = latitude (north or south) of the launching base, $0° \leqslant \varphi \leqslant 90°$;

φ_P = latitude of the perigee of the target orbit, $0° \leqslant \varphi_P \leqslant 90°$, $\sin \varphi_P = |\sin i \sin \omega|$.

h = altitude above the ground;

R = equatorial radius of the planet investigated;

E = equatorial velocity of rotation of the planet investigated (= 465 m/s in the case of Earth);

K = circular low velocity of the planet investigated (= 7,905 m/s in the case of Earth).

$C(t) = \int_0^t |\vec{\gamma}|\, dt + \Sigma\, |\vec{\Delta V}|$ = characteristic velocity ($\vec{\gamma}$ designating the accelerations of thrust and $\vec{\Delta V}$ designating the impulses).

II.3 Results /5

II.3.1 General Comments

I) On the parameter Ω_B.

It is clear that it is always possible to obtain Ω_B, right ascension of the ascending equatorial node, as desired, for an identical cost by selecting the suitable launching time. Nevertheless, if the rotation of the planet is slow (Mercury, Venus), it will be possible for this reason to be led to wait, if necessary, for several months. If this is not possible, then launch can take place without waiting. The problem is, in this case, similar to the one for launching from a polar base.

The parameter Ω_B will therefore no longer become a factor in the study.

II) On ascents into parking orbit.

The ascents into a more or less low parking orbit are a frequent first stage of space missions either because they appear during investigation of the optimum process or because it is desired to make a transition to such an orbit for various reasons (technical convenience, precision of subsequent stages, etc...).

In each one of these two cases the minimization of the cost of the mission leads to the selection of as low orbits as possible (the optimum altitude being a function of the atmosphere). On the other hand, it is not obligatory for the ascent into the parking orbit to be fast and it may be that the best parking orbit will have a plane which does not intersect the parallel of the launching site. However, in consideration of the infrequency of this case and the complication of the maneuvers to which it gives rise, we shall assume it as excluded. Under these conditions, the optimum ascent into parking orbit is a fast ascent phasing immediately into the crossing of the atmosphere.

We shall state for this first intermediate orbit whose optimum shape is circular as follows:

semimajor axis: $a_1 = R + h_1$;

eccentricity: $e_1 = 0$.

inclination: i_1 (with $\varphi \leqslant i_1 \leqslant 180° - \varphi$ taking the comment made above into consideration.

The characteristic velocity C_1 required for this first phase is independent of the latitude of the launching site. The purely geometric study which arrives at a slight disadvantage for equatorial sites is indeed exactly counterbalanced by the effect of the planet's flattening (it is enough, moreover, to examine the question from the energy angle in the axis linked to the planet in rotation: all launching sites located at the same level correspond to the same initial energy. The effect of the altitude of the launching site on the theoretical characteristic velocity is, moreover, quite little: 1.24 m/s for 1 km in the case of the Earth. On the other hand, this altitude clearly can have a great deal of effect on the cost of crossing the atmosphere).

C_1, theoretical characteristic velocity (i.e., not taking into account the cost of crossing the atmosphere) of putting into a low parking orbit is only the function of i_1 and h_1:

$$C_1 = \sqrt{K^2 + E^2 - 2\,EK\cos i_1} + \frac{h_1 K}{2\,R}$$

or in the case of the Earth:

$$C_1 = 7\,905 \text{ m/s} - 465 \text{ m/s} \cos i_1 + 14 \text{ m/s} \sin^2 i_1 + \left[\frac{h_1}{100 \text{ km}}\right] 62 \text{ m/s}.$$

If the asymmetries of the terrestrial potential are taken into account, this formula is hardly modified:

$$C_1 = 7\,911 \text{ m/s} - 465 \text{ m/s} \cos i_1 + 10 \text{ m/s} \sin^2 i_1 + \left[\frac{h_1}{100 \text{ km}}\right] 62 \text{ m/s}$$

(h_1 is in this case the altitude of passage above the equator).

The term at h_1 is partly recovered in the subsequent phases of the ascent.

The optimum altitude h_1 is a function of the atmosphere, the ascent under study, the aerodynamic characteristics of the rocket under study and the retention time in the parking orbit (a time which is understandably advantageous to keep short). In the case of the Earth, this altitude is almost always included between 100 and 200 km.

Comment: In order to fully take into account the effect of crossing the atmosphere, it can be said that everything takes place (from the viewpoint of characteristic velocity) as the first of one or more stages used for this crossing produced a characteristic velocity equal to:

$$C_0 = V_h + \frac{V_v^2}{2(K + V_h)} + \frac{h_0 K}{R}$$

with: V_h = horizontal velocity (with respect to the ground) of "atmospheric exit";

V_v = vertical velocity of "atmospheric exit";

h_0 = altitude of "atmospheric exit".

(in the case of the Earth $\frac{h_0 K}{R} = \frac{h_0}{100 \text{ km}} \cdot 124$ m/s; $h_0 = 40$ at 70 or even 100 km according to V_v; V_v is assumed on the order of 0 to 1 km/sec and V_h of 0 to 7 km/sec).

It is the quantity C_0, practically independent of the orientation, which is suitable for optimizing during the crossing of the atmosphere. This generally leads to " exits" slightly inclined to the horizontal. The subsequent stages should then supply the characteristic velocity $C_1 - C_0$ in order to complete the putting into parking orbit and C_1 possesses in this case a precise physical direction.

II.3.2 Results in the Case where $\varphi \leqslant i_B \leqslant 180° - \varphi$

This is the most simple and economical case.

We shall assume that one of the following hypotheses is confirmed:

Either $\frac{E}{K} \leqslant 0.1$, suitable for the planets of Mercury to Mars $\left(\frac{E}{K} \text{ Earth} = \frac{1}{17}.\right)$

Or $\varphi = 0$ and $\frac{E}{K} \leqslant 0.5$, suitable for all the planets of the solar system.

$\left(\frac{E}{K}\right.$, maximum $= \frac{E}{K}$ Saturn $= 0.41)$.

The study of the optimum can then be made completely if ω_B is open or if $\omega_B = 0°$ or $180°$ (values which are precisely the optimum values when a_B, e_B and i_B are fixed) and it can be done simply if $\omega_B \neq 0°$ and $180°$ on condition of allowing, with respect to the strict optimum, a loss (in characteristic) velocity) less than:

$$\frac{E^2}{2K} \sin^2 i_B \sin^2 \omega_B \left[1 - \sqrt{\frac{R + A_B}{2 A_B}}\right]$$

or at the maximum 4 m/s in the case of the Earth.

The optimum ascents into orbit are then of three possible types.

These three types begin by a putting into a low parking orbit (this not being the case for the strict optimum if $\omega_B \neq 0°$ and 180°). This stage can occasionally be skipped but it can always be done provided that a minimum increase in cost be accepted (this being zero in the absence of atmosphere). /6

We shall call these three types of optimum ascent the "direct ascent", "ascent through infinity" and the ascent "through infinity and then through the atmosphere".

II.3.2.1 The Direct Ascent

The direct ascent (Fig. 2) includes, after putting into a low parking orbit 0_1, two impulses at I_1, then at I_2 with an intermediate orbit 0_2.

I_2 is generally very close to apogee A_B of the target orbit and I_1 is generally quasi-diametrically opposed to I_2.

The optimization of the inclination i_1 of 0_1 and of the positions of I_1 and I_2 is quite complicated. Nevertheless, if there is granted, with respect to the strict optimum, a loss which instead of being limited to:

$$\frac{E^2}{2K} \sin^2 i_B \sin^2 \omega_B \left[1 - \sqrt{\frac{A_B + R}{2 A_B}}\right],$$

can rise to

$$\frac{E^2}{2K} \rho \sin^2 i_B \sin^2 \omega_B$$

with ρ =

$$1 + \sqrt{\frac{A_B + R}{2 A_B}} \left[\frac{A_B}{R} - 1 - A_B \sqrt{\frac{A_B + P_B}{RP_B (A_B + R)}}\right]$$

(ρ, which is always positive, remains less than 1 if $A_B \leqslant 3.21$ R and than 2 if $A_B \leqslant 5.27$ R), it is then always possible to place the point I_2 of Figure 2 at the apogee A_B of the target orbit and the point I_1 diametrically opposed to I_2. I_1 and I_2 are in this case the perigee and the apogee of the intermediate orbit 0_2. The "practical optimum ascent is thus obtained (Fig. 3). The transfer from 0_1 to 0_B is of the "generalized Hohmann" type [2]. The plane of 0_1 contains the major axis of 0_B. All that remains is to optimize the values of i_1 and i_2 (cf. the annex).

If $\frac{E}{KR}\sqrt{\frac{A_B(A_B+R)}{2}} < 1$ (or if $A_B < 23.5$ R in the case of Earth) it is possible to develop the optimum value of i_1 as well as i_2 as a function of the components of 0_B:

Let us grant:

$$\delta_B = \rho \frac{E}{K} \sin i_B \left[1 - (\rho - 1)\frac{E}{K}\cos i_B\right] +$$

$$+ \text{ order } \left[\frac{E \sin i_B}{KR}\sqrt{\frac{A_B(A_B+R)}{2}}\right]^3$$

ρ having the value defined in the preceding paragraph. There is produced (for the "optimum practical ascent"):

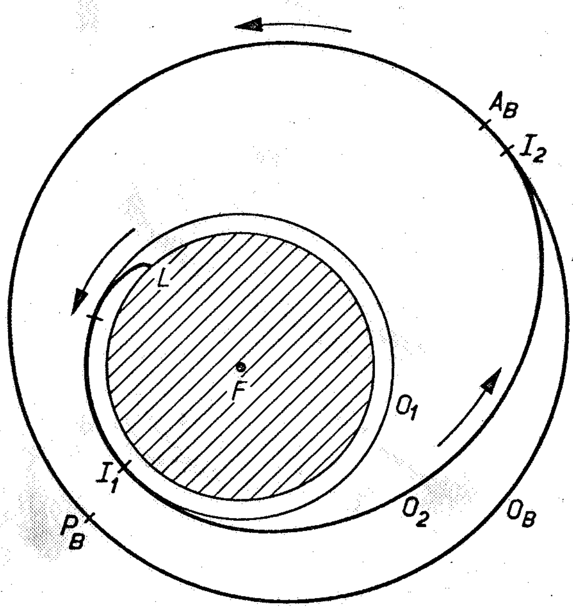

Fig. 2. Direct Ascent. The Orbits 0_1, 0_2 and 0_B are in Generally very closely Related Planes.

$$i_1 = \max[\varphi;\, i_B - \delta_B \cos^2 \omega_B] \leqslant i_B$$

$$i_2 = i_1 + \frac{i_B - i_1}{\rho}\left[1 - \sqrt{\frac{R + A_B}{2 A_B}}\right] : i_1 \leqslant i_2 \leqslant i_B$$

δ_B is on the order of several degrees in the common terrestrial cases. The total characteristic velocity of the "optimum practical ascent" is:

C_B practical =

$$K\left[\sqrt{\frac{2A_B}{A_B+R}}\left(1 - \frac{R}{A_B}\right) + \sqrt{\frac{R}{A_B}}\,(1 - e_B)\right]$$

$$+ \frac{h_1 K}{R}\left[1 - \frac{(3R + A_B)\sqrt{A_B}}{\sqrt{2}\,(A_B+R)^3}\right] - E\cos i_B +$$

$$+ \frac{E^2 \sin^2 i_B}{2K}\left[1 + \rho\left[\left(\cos\omega_B - \frac{i_B - i_1}{\delta \cos \omega_B}\right)^2 - \cos^2\omega_B\right]\right] +$$

$$+ \text{ order } \left[\frac{E^3}{2K^2}(\rho^2+1)\sin^2 i_B\right]$$

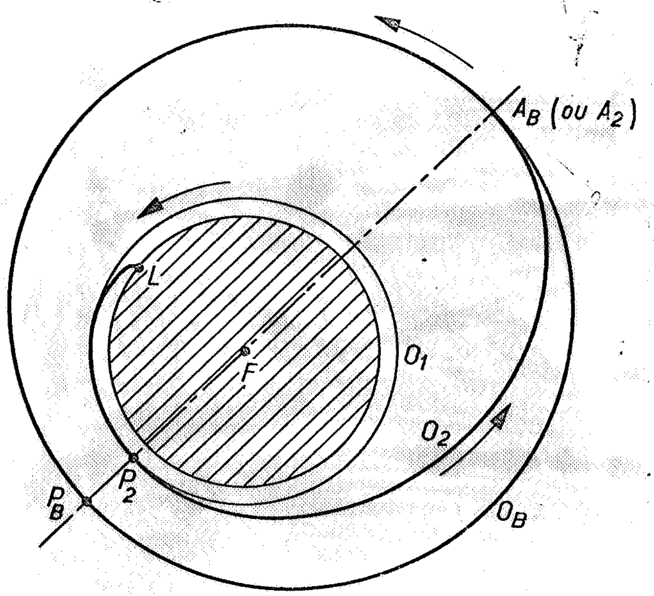

Fig. 3 Optimal Practical Ascent. The Orbits O_1, O_2 and O_B are "Direct Coaxial" ones and are in Generally Very Closely Related Planes ($i_1 \leqslant i_2 \leqslant i_B$); b, A_B (or A_2).

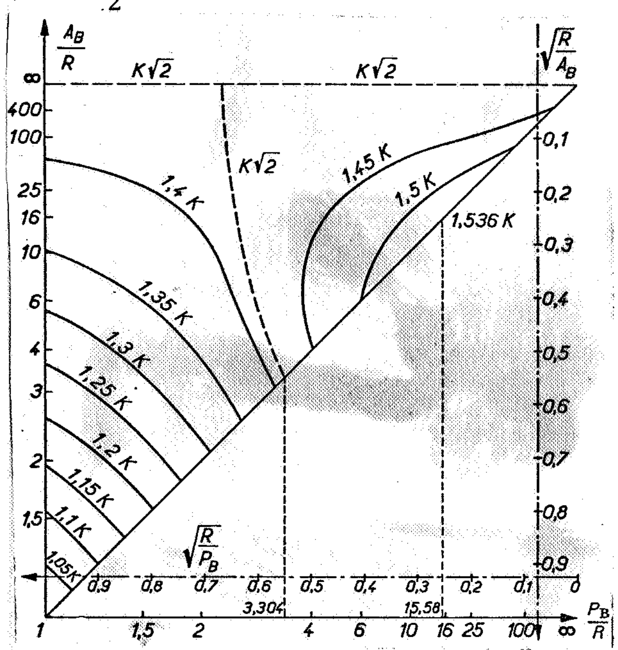

Fig. 4 Value of the Term:

$$K\left[\sqrt{\frac{2A}{R+A_B}}\left(1-\frac{R}{A_B}\right)+\sqrt{\frac{R}{A_B}}\,(1-e_B)\right]$$

as a function of $\frac{PB}{R}$ and $\frac{A_B}{R}$.

It is appropriate to comment that for the "optimum practical ascent" and, moreover, for the corresponding direct ascent and strict optimum the following is true: /7

$$C_B = K\left[\sqrt{\frac{2A_B}{A_B+R}}\left(1-\frac{R}{A_B}\right)+\sqrt{\frac{R}{A_B}}\,(1-e_B)\right]+ + \frac{h_1 K}{R}\left[1-\frac{(3R+A_B)\sqrt{A_B}}{\sqrt{2(R+A_B)^3}}\right] - E\cos i_B + \varepsilon$$

with

$$|\varepsilon| \leqslant \frac{E^2\sin^2 i_B}{2K}, \text{ si } A_B \leqslant 5{,}27\ R$$

and

$$|\varepsilon| \leqslant \frac{E^2\sin^2 i_B}{2K}\left(\frac{A_B}{R\sqrt{2}}-2{,}72\right), \text{ si } A_B \geqslant 5{,}27\ R$$

if $A_B \geqslant 5.27$ R.

($\frac{E^2}{2K}$ = 13.7 m/s in the case of Earth)

The parameters φ and ω_B are a very small factor in the cost of the ascent. h_1 is a small factor: in the case of the Earth the term at h_1 is:

$$\left(\frac{h_1}{100\text{ km}}\right)\cdot 5\text{ m/s, if } A_B = 2R$$

$$\left(\frac{h_1}{100\text{ km}}\right)\cdot 23\text{ m/s if } A_B = 8R$$

$$\left(\frac{h_1}{100\text{ km}}\right)\cdot 36\text{ m/s if } A_B \sim +\infty.$$

The parameter i_B is more of a factor: E cos i_B gives only 465 m/s cos i_B in the case of the Earth (much more, however, for the large planets). It is clearly the first term which is the most important. It is always included between K and 1.536 K.

Figure 4 shows its variations as a function of P_B and A_B. It may be especially noted that it is always less than $K\sqrt{2}$ if $P_B \leqslant \frac{9}{4}$ R and more than $K\sqrt{2}$ if $P_B \geqslant 3.304$ R. The orbits with low perigee, even the very eccentric ones, are more accessible than the far orbits.

II.3.2.2 The Ascent "Through Infinity".

This second method of optimum ascent is quite rarely the best one and the saving which it allows with respect to the "direct ascent" never exceeds K.0.043 + E $(\cos \varphi - \cos i_B)$ which is little in the common terrestrial cases.

Figure 5 shows it in a case where $\varphi_p \leqslant \varphi$. There is then successively:

1. A putting into waiting orbit 0_1, with inclination $i_1 = \varphi$ (launching towards the East).

2. Three successive impulses at P_2, A_2 and P_B with two very elongate intermediate ellipses 0_2 and 0_3 with common apogee A_2. The orbit 0_2 is tangent to 0_1 at its perigee P_2 (therefore $i_2 = i_1 = \varphi$), the orbit 0_3 is tangent to 0_B at their common perigee P_B (therefore $i_3 = i_B$). The three orbits 0_2, 0_3, 0_B have the same direction of major axis (determining P_2). The impulse at A_2 is very slight.

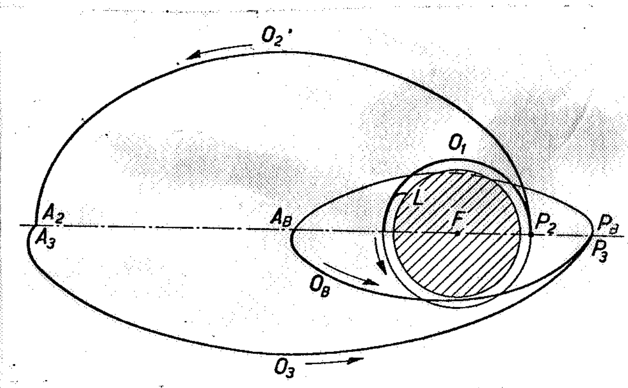

Fig. 5 Ascent "Through Infinity"

The characteristic velocity of the ascent is:

$$C_\infty = \sqrt{K^2 + E^2 - 2\,EK\cos\varphi} + K(\sqrt{2} - 1) + \frac{h_1 K}{2\,R}(2 - \sqrt{2}) + K\sqrt{\frac{R}{P_B}}(\sqrt{2} - \sqrt{1 + e_B}).$$

It is advisable to add to C_∞ the loss owing to the unavoidable non-movement apart at infinity of A_2. This loss, if A_2 is major, is:

$$\delta C_\infty = \frac{K\sqrt{2\,R}}{A_2}\left[\sqrt{P_B + R - 2\cos\alpha\sqrt{RP_B}} - \frac{\sqrt{R}}{2} - \frac{\sqrt{P_B}}{2}\right]$$

α being the angle of the oriented planes of 0_2 and 0_3 (the arrangement of Figure 5 with an impulse at A_2 - merged with A_3 - corresponds practically with the optimum when the duration of ascent into orbit or even the maximum movement apart is fixed).

For $A_2 = 60\, R\, \delta C\infty$ is, in the case of Earth, on the order of some tens of m/s.

If $\varphi_p > \varphi$ the arrangement of Figure 5 cannot be realized. It could theoretically be possible to obtain the ascent into orbit for the same cost by interposing between 0_2 and 0_3 a very far intermediate orbit $0_2'$ which would allow placing 0_3 in the desired orientation (0_3 with same plane and major axis as 0_B) but, given the very slow decrease of circular velocities at great distance, this solution offers no practical advantage for planets with slow rotation (Mercury to Mars). It would be worthwhile in this case to somewhat modify the orientation of 0_1 ($\varphi \leqslant i_1 \leqslant \varphi_p$) and the arrival point on 0_B. $C\,\infty$ is then at the increased maximum of the quantity: /8

$$\sqrt{K^2 + E^2 - 2\,EK\cos\varphi_B} - \sqrt{K^2 + E^2 - 2\,EK\cos\varphi}$$

(or practically $E\,(\cos\varphi - \cos\varphi_p)$.

II.3.2.3 The Ascent "Through Infinity then Through the Atmosphere".

This third method of optimum ascent is also rather infrequently the best and the saving it allows with respect to the "direct ascent" never exceeds

$$E\,(\cos\varphi - \cos i_B) - K\left[\sqrt{2} - \sqrt{\frac{2\,A_B}{A_B + R}}\right],$$

which is very little in the common terrestrial cases. What is more, it requires that atmospheric brakings be possible (with a lift-drag ratio which can conveniently be zero).

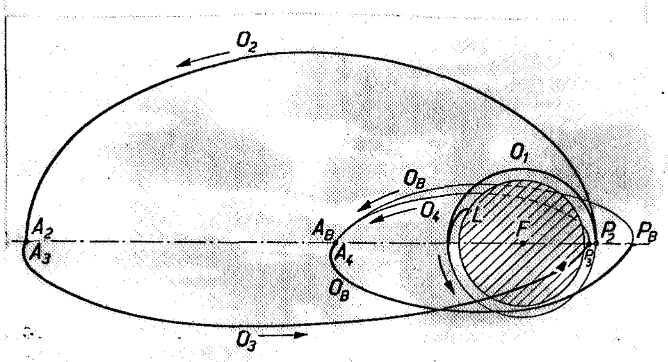

Fig. 6 Ascent "Through Infinity then Through the Atmosphere.

Figure 6 shows it in a case where $\varphi_p \leqslant \varphi$. There is in this case successively:

1. A putting into parking orbit 0_1 with inclination $i_1 = \varphi$ (launching towards the East).

2. Three successive intermediate orbits 0_2, 0_3 and 0_4 separated by: a tangential impulse at P_2 (therefore $i_2 = i_1 = \varphi$), an infinitely small impulse at A_2, very far common apogee of 0_2 and 0_3, an atmospheric braking at P_3 (merged with P_4), finally a tangential impulse at A_B (or A_4). i_4 is equal to i_B but owing to the rotation of the planet a slightly different i_3 should be chosen.

The characteristic velocity of the ascent is:

$$C_{\infty\,atm} = \sqrt{K^2 + E^2 - 2\,EK\cos\varphi} + K(\sqrt{2} - 1) + \\ + \frac{h_1 K}{2\,R}(2 - \sqrt{2}) + K\left[\sqrt{\frac{R}{A_B}}(1 - e_B) - \sqrt{\frac{2\,RP_3}{A_B(A_B + P_3)}}\right].$$

The comments made on the subject of ascents "through infinity" can be repeated here. Especially concerned is the loss owing to the non-moving apart at infinity of A_2 is (for A_2 major):

$$\delta C_{\infty\,atm} = \frac{KR\sqrt{2}}{2\,A_2}\left[4\sin\frac{\alpha}{2} - 1\right]$$

(α being again the angle of the oriented planes of 0_2 and 0_3).

II.3.2.4 For the sake of completeness, it is advisable to mention a very exceptional case of optimum ascent only occurring when $\varphi \neq 0°$ and when atmospheric brakings (and hence the ascent "through infinity then through the atmosphere") are impossible. This case uses, between the parking orbit 0_1 for which $i_1 = \varphi$ and the target orbit 0_B, a "three-impulse" transfer similar to the one described in reference 2 (and even strictly identical if $\omega_B = 0°$ or $180°$).

II.3.3 Results in the Cases $i_B < \varphi$ and $i_B > 180°-\varphi$

These cases are at the same time the most complicated and the most costly.

We shall still assume $\frac{E}{K} \leqslant 0.1$ and we shall additionally assume, in order to avoid problems of sign, that the launching site is in the northern hemisphere.

The optimum is then a function of the will to make a transition or not to a low parking orbit.

II.3.3.1 Case where it is Desired to make a Transition to a low Waiting Orbit.

The optimum method of ascent can in this case be "through infinity" or even "through infinity then through the atmosphere", cases similar to those obtained for $\varphi \leqslant i_B \leqslant 180°-\varphi$. (II.3.2.2 and II.3.2.3), but the saving realized with respect to a "direct ascent" can be very great here.

If the parameter ω_B is open, or even if $\sin\omega_B = 0$ ($0°$ and $180°$ are the optimum values of ω_B when a_B e_B and i_B are fixed), the study can be made completely. In addition to the two cases above, the following three methods may be met:

I. The direct ascent, similar to the one found in paragraph II.3.2.1. In this case, I_1 and I_2 are the apexes of the major axis and the equatorial nodes of 0_2. The value of i_1 is: φ if $i_B < \varphi$, a function of the study performed in the annex if $i_B > 180°-\varphi$.

II. The "three-nodal ascent" (Fig. 7). Transition is made from 0_1 to 0_B by a "three-impulse transfer" of reference 2 (the three impulses are here at equatorial nodes).

$$i_1 = \varphi \text{ if } i_B < \varphi.$$

$$i_1 = \varphi \text{ or } 180° - \varphi \quad \text{if } i_B > 180° - \varphi.$$

III. The "four nodal ascent", version with a point A_2 at finite distance of the ascent "through infinity then through the atmosphere". (II.3.2.3) The orbit 0_2 is not in this case in the plane of 0_1 and the orbit 0_4 is not in the one for 0_B. The corresponding impulses, at P_2 and A_B, are not tangential. /9

Here again: $i_1 = \varphi$ if $i_B < \varphi$ and $i_1 = \varphi$ or $180°-\varphi$ if $i_B > 180°-\varphi$.

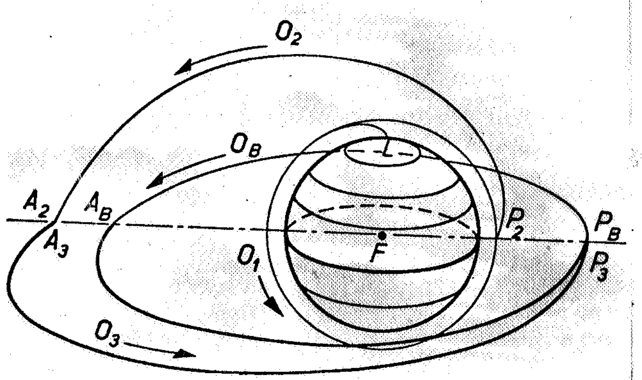

Fig. 7 "Three Nodal Ascent". The Line of Apsides is also the Line of the Equatorial Nodes.

It can be noted that in all cases: $i_1 = \varphi$ if $i_B \leqslant \varphi$ (this is, moreover, still true if $\omega_B \neq 0°$ and $180°$, cases for which study is not yet complete).

II.3.3.2 Case where it is Open for Transition to a Low Parking Orbit

In this case there are again found, of course, ascents "through infinity" and ascents "through infinity then through the atmosphere" when the target orbit is far or even very eccentric. The common cases, however, are not solved.

Nevertheless, if we state:

1° $\delta i = \varphi - i_B$ si $i_B < \varphi$.
$\quad = i_B - (180° - \varphi)$ si $i_B > 180° - \varphi$.

2° $\dfrac{a_B - R}{a_B} = \delta a$; (donc $\delta a \geqslant e_B$)

3° $\varepsilon = \max(\delta i, \delta a)$.

We can write:

If ε is minor and it is not possible to use atmospheric brakings the best of the two-impulse solutions (therefore with one immediate impulse, generally oblique and the greater of the two, and one other impulse to the target orbit) represents, if not the optimum, which is quite frequent, at least a solution very close to the optimum (the loss, in characteristic velocity, is never greater than the order $K\varepsilon^2$ and probably even than the order $K\varepsilon^3$).

Assuming the launching site to be in the northern hemisphere, the characteristic velocity of the ascent is then:

If it is true $e_B \sin \omega_B = f$.

It follows that:

First case:

$$f + \delta i \frac{\sqrt{3}}{3} \geqslant \delta a$$

$$C_B = K\left[1 + \frac{f}{2} + \frac{\delta i}{2}\sqrt{3}\right] \pm E \cos \varphi + \text{order}\left[\varepsilon^2 K; \frac{E^2 \sin^2 \varphi}{2\,K}\right]$$

(this expression is not exact in the very exceptional case where

$$90^\circ < \omega_B < 115^\circ{,}3 \text{ et } \frac{e_B}{\delta a} > 0{,}97572).$$

Second case:

$$f + \delta i \frac{\sqrt{3}}{3} \leqslant \delta a$$

$$C_B = K\left[1 + f - \frac{\delta a}{2} + \sqrt{\delta i^2 + (\delta a - f)^2}\right] \pm E \cos \varphi + \text{order}\left[\varepsilon^2 K; \frac{E^2 \sin^2 \varphi}{2\,K}\right].$$

The quantity $\pm$ E $\cos \varphi$ should be taken as equal to $+E \cos \varphi$ if $i_B > 180° - \varphi$ and to $-E \cos \varphi$ if $i_B < \varphi$.

It is possible to compare these values of C_B to those obtained when there is an obligation to make transition to a parking orbit (the ascent is generally of the "direct ascent" type). We shall disregard the effect of altitude on this parking orbit.

It follows that:

First case:

$$f^2 - \delta i^2 + \frac{2}{2}\,\delta i|f|\sqrt{3} \geqslant \delta a^2$$

$$C_B = \left[1 + \frac{|f|}{2} + \frac{\delta i}{2}\sqrt{3}\right] \pm E \cos \varphi + \text{order}\left[\varepsilon^2 K; \frac{E^2 \sin^2 \varphi}{2\,K}\right].$$

Second case: $$f^2 - \delta i^2 + \frac{2}{3}\delta i\,|f|\sqrt{3} \leqslant \delta a^2$$

$$C_B = K(1 + X) \pm E\cos\varphi + \text{order}\left[\varepsilon^2 K; \frac{E^2\sin^2\varphi}{2K}\right]$$

order

with:

1. $$X \geqslant \sqrt{\frac{\delta a^2}{4} + \delta i^2}$$

2. $$f^2(4X^2 - \delta a^2) = \left(X^2 - \frac{\delta a^2}{4} - \delta i^2\right)(4X^2 + 3\delta a^2).$$

The refusal to make transition through a parking orbit can provide considerable saving: thus, in the extreme case where:

$$\delta i = \frac{14\sqrt{3}}{23}\delta a \text{ and } f = -\delta a$$

(therefore $e_B = \delta a$; $\omega_B = 270°$) the terms at K are:

$$K\left[1 + \frac{35}{46}\delta a\right]$$

if a transition is not made through a parking orbit and $K\left[1 + \frac{65}{46}\delta a\right]$ if a transition is made.

If, in addition to the possibility of not passing through a low parking orbit, atmospheric braking (with a lift-drag ratio of zero) can be carried out, it is still possible to make substantial savings (the ascents of this type include one or two impulses following the single atmospheric braking).

This, for example, in the case $e_B = 0$ and $a_B = R$, with the order $\left(\varepsilon^2 K \text{ et } \frac{E^2\sin^2\varphi}{2K}\right)$

$C_B = K(1 + \delta i) \pm E\cos\varphi$ if transition made through a low parking orbit (with or without use of atmospheric braking).

$C_B = K\left(1 + \delta i\frac{\sqrt{3}}{2}\right) \pm E\cos\varphi$ if transition not made through a parking orbit and if atmospheric braking not used.

$C_B = K\left(1 + \delta i\frac{\sqrt{7}}{4}\right) \pm E\cos\varphi$ if transition not made through a parking orbit and if atmospheric braking not used:

$$\left(\frac{\sqrt{7}}{4} = 0{,}661 < \frac{\sqrt{3}}{2} = 0{,}866 < 1\right).$$

We shall not further study ascents of this type which still remain to a great extent unexplored. It is true that they correspond to worst case in which the target orbit does not intersect the parallel of the launching site.

II.4 Discussion

In a special determined case, it is clearly a requirement to begin by investigating which is the optimum method of ascent.

Hence: /10

First case: Atmospheric braking can be used and the lift-drag ratio of the rocket investigated is infinite.

It is then possible to develop gradually without cost in the whole atmosphere without losing velocity and the various launching sites become equivalent to the best among them, namely the equatorial site. The discussion is reduced to that of the case $\varphi = 0°$ (for which a non-zero lift-drag ratio is useless).

Second case: the lift-drag ratio of the rocket investigated is zero and $\varphi \leqslant i_B \leqslant 180° - \varphi$.

It is then possible to make transition to a low intermediate parking orbit for a very slight increase in cost.

The optimum ascent is of the "direct ascent" type or even "ascent through infinity then through the atmosphere" (the latter requiring an atmospheric braking).

The comparison of the two last methods leads practically to:

$P_B < 4R\left(1 + \frac{R}{A_B}\right)$ the "∞ atm" method is better than the one "through infinity".

$P_B > 4R\left(1 + \frac{R}{A_B}\right)$ this is the opposite.

There remains the comparison of the better of these two methods with the "direct ascent". Thus, for example:

If: $A_B \leqslant R\left[\frac{K\sqrt{2}}{2E(\cos\varphi - \cos i_B)} - \frac{3}{4}\right]$.

Or if: $A_B \leqslant R\left[\frac{12}{\cos\varphi - \cos i_B} - \frac{3}{4}\right]$ in the case of the Earth the "direct ascent" is better than the ascent "through infinity then through the atmosphere".

Likewise, in the case of the Earth, if

$$\frac{1}{A_B} \geqslant \frac{1}{R}\left[\frac{1}{11{,}938} + \frac{\cos\varphi - \cos i_B}{2}\left[\frac{1}{5{,}04} - \frac{1}{11{,}938}\right]\right]$$

the "direct ascent" is better than the ascent "through infinity" (the limit on A_B varying from 11.938 R to 5.04 R).

From the practical viewpoint, it can be said that the "direct ascent" represents the customary solution for the optimum ascent into orbit, a solution offering furthermore the advantages of simplicity, convenience and speed.

Third case: $i_B < \varphi$ or even $i_B > 180°-\varphi$.

The optimization of the ascent into orbit is in this case a strong function of the capability of performing atmospheric braking and of the need to make transition or not to an intermediate parking orbit (cf. Fig. 10 in the following Chapter II.5).

Figures 8 and 9 show the results of the discussion as a function of the parameters:

$$|i_B - i_1| \text{ and } \frac{A_B}{R_1} \text{ on one hand, } |i_B - i_1| \text{ and } \frac{P_B}{R_1}$$

on the other hand in the case where atmospheric braking is not used, where it is desired to make transition to an intermediate parking orbit with inclination i_1 and with radius $R_1 = a_1 = R + h_1$ and where the parameter ω_B is either open or equal to 0° or 180°. The optimum ascents are the direct (here two-nodal) or "three-nodal" (cf. Fig. 7) or "through infinity". (Keep in mind that the optimum value of i_1 is always φ if $i_B \leqslant \varphi$).

In doubtful cases, completion of the discussion is a function of $\frac{P_B}{R_1}$.

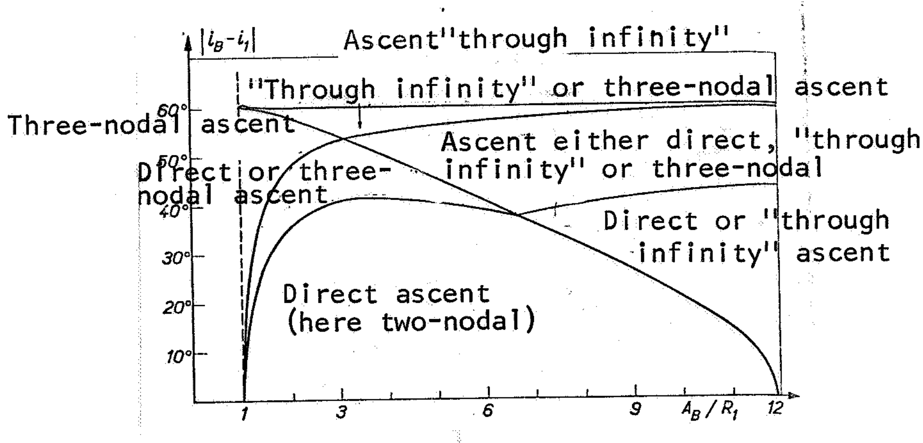

Fig. 8. Discussion of the Optimum Method of Ascent as a Function of $\frac{A_B}{R_1}\left(=\frac{A_B}{a_1}\right)$ and of $|i_B - i_1|$.

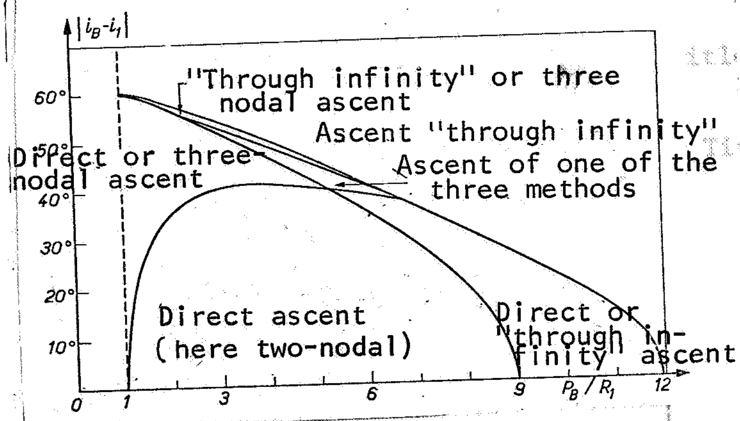

Fig. 9. Discussion of the Optimum Mode of Ascent as a Function of $\frac{P_B}{R_1}\left(=\frac{P_B}{a_1}\right)$ and of $|i_B - i_1|$.

In doubtful cases the completion of the discussion is a function of $\frac{A_B}{R_1}$.

If $e_B = 0$, or if $A_B = P_B$ the three-nodal zone is with maximum extension.

General comment: if

$$P_B \geqslant 9\,R\left[1+\frac{3{,}9\,R}{A_B}\right]$$

the optimum ascent is always of the "through infinity" type no matter what the values of i_B, φ and ω_B.

11.5 Comparison of the Various Launching Sites (4")

We have seen from the space viewpoint, that the advantage of a launching site was not a function of its longitude and had little to do with its altitude (except insofar as the cost of crossing the atmosphere was concerned). The essential components of this problem then concerns the latitude and the one or more launching sectors.

For practical purposes, unless acrobatics of the angled launch variety /11 are contempleted (useable if the second stage, which is lighter, is not reduced to the same launching sector as the first one) or even if consideration be given to quasi-vertical launching (useable profitably by bases having available only of either a very small sector or an entirely western sector), the advantage of a launching base is defined by the aggregate of inclinations i_1 of low parking orbit which are directly accessible from the base in question (with, of course, $\varphi \leqslant i_1 \leqslant 180°-\varphi$). In this way, a base having only the north (or south) azimuth available is approximately equivalent to a polar azimuth.

The inclinations i_1 greater than 90°, corresponding to western launching azimuths require a more costly putting in orbit than those for which $i_1 \leqslant 90°$ (owing to the rotation of the planet). They are therefore less advantageous and little used. It should nevertheless be noted that they can fulfill some specific missions whether the measurement sought after is a function of the direction of movement of the satellite (e.g., measurement of upper atmospheric winds) or whether use is made of secondary effects proper to the target orbit such as those resulting from perturbations caused by equatorial bulge, perturbations allowing for example, the maintenance without cost of the node, or perigee, in a direction almost fixed with respect to the sun on the condition that a_B, e_B and i_B are appropriately selected (for the node, if the orbit is low, i_B is close to 100°).

But, outside of these cases, the eastern launch azimuths and inclinations i_B less or equal to 90° are always preferable.

On the other hand, two launch azimuths, symmetrical with respect to the east-west line, lead to the same inclination i_1 of low parking orbit. They only provide, therefore, an advantage of manageability (two hours daily when launching is possible instead of one).

Therefore, in practice, for a given latitude φ the best base is the one having available a launching sector which encompasses the whole northeast quadrant or even the whole southeast quadrant and, so far as φ is concerned, the best base is the one closest to the equator.

It can therefore be concluded by stating that the Kourou base of our fellowcountrymen in Guiana, at 5° north latitude, is, from the space viewpoint, the best or at least one of the best in the world. It has available indeed a launching sector of at least 140° from the azimuth of 330° (or-30°) to that of 110°, passing through the north and east (allowing directly reaching the inclinations i_1 from 5° to 117°).

Figure 10 illustrates the importance of position and launching sector of the launch bases. It shows the characteristic velocity of ascent into orbit in the extreme case in which the target orbit is equatorial and low circular (altitude 130 km) as a function of the latitude of the "all azimuths" launching base, or, which amounts to almost the same thing, as a function of the minimum inclination i_1 which is directly accessible from a given base (if it is possible to launch towards the east i_1 mini = φ, if not i_1 mini > φ).

Of course, this importance can vary greatly depending on the target orbit. Thus, the cost of installation on close polar orbit ($A_B \leqslant 5$ R) is almost the same for all bases having a north or south launching azimuths.

One mean case is supplied by far equatorial orbits. Here, the cost of installation on geostationary orbit varies from 11.45 km/sec (equatorial base, launch towards the east, direct ascent) to 12.45 km/sec (polar base, ascent "through infinity" and even to 12.91 km/sec (equatorial base having available only a launching azimuth of 270°: towards the west, ascent "through infinity").

X = φ if launch can be made towards the east from the base selected. Otherwise X = i_1 minimum accessible (hence, in this case X > φ).

The dotted line curves can only be used if there is no requirement to make transition to a low intermediate parking orbit.

1. The three upper curves relate to the case where atmospheric braking cannot be used; the three others in the case where it

is possible.

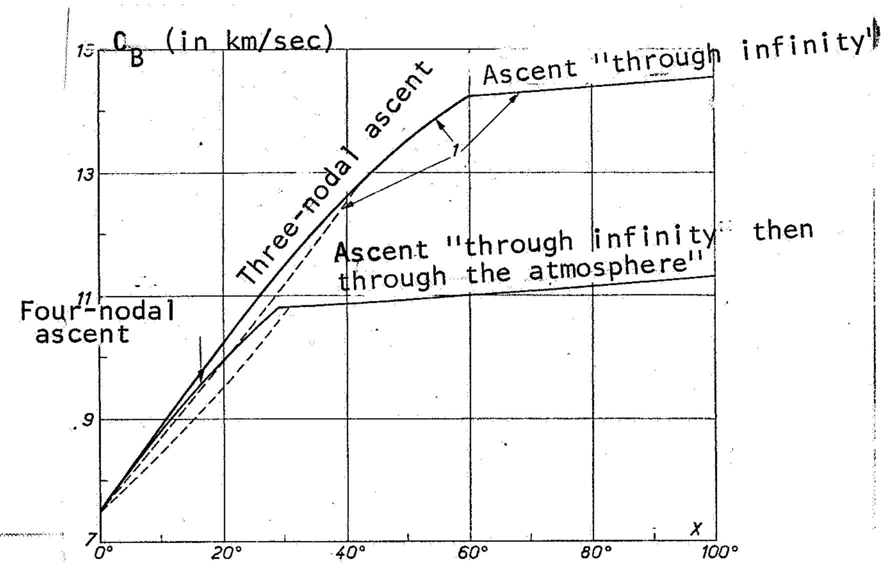

Fig. 10 Cost C_B (in km/sec) of Ascent into Low Circular Equatorial Orbit (h_B = 130 km) as a Function of X.

CONCLUSION

The study of the optimization of the ascent into orbit when the target orbit is elliptical and wholly determined is still very incomplete.

The results already obtained allows nevertheless the statement that the advantage of a launching base is practically determined by the aggregate of the inclinations of low parking orbit directly accessible from the given base, an aggregate which is, of course, a function of the latitude of the base and of the one or more launching sectors which are available.

In the best case at hand (Guiana base) these directly accessible inclinations go from 5° to 117°.

For these inclinations the optimum method of ascent into orbit is most often the "direct ascent" which has three rocket phases (one ascent into low intermediate parking orbit, then a transfer similar to the customary perigee-apogee transfer). Nevertheless, if the target orbit is far or very eccentric, "through infinity" methods can become optimum.

With respect to the other inclinations, which are much more unfavorable, the study has remained very incomplete. It is much a function of the need to make a transition or not to a low intermediate parking orbit. The optimum orbits are classified into many more or less complicated methods. It would be advantageous to study these cases somewhat more in depth, but in practice, of course, it would be sought to avoid them when selecting the suitable launching base.

ANNEX /12

Optimization of the values of i_1 and i_2 in the "practical optimum ascent" (Fig. 3).

The investigation will only be made in the case in which $\omega_B = 0°$ or 180° (case in which the "optimum practical ascent" corresponds to the strict optimum). The investigation would be similarly performed if $\sin \omega_B \neq 0$.

1. Investigation for $\varphi = 0°$.

Since the two first propellant phases are carried out at the same place, they can be carried out together. There are therefore only two distinct propellant phases.

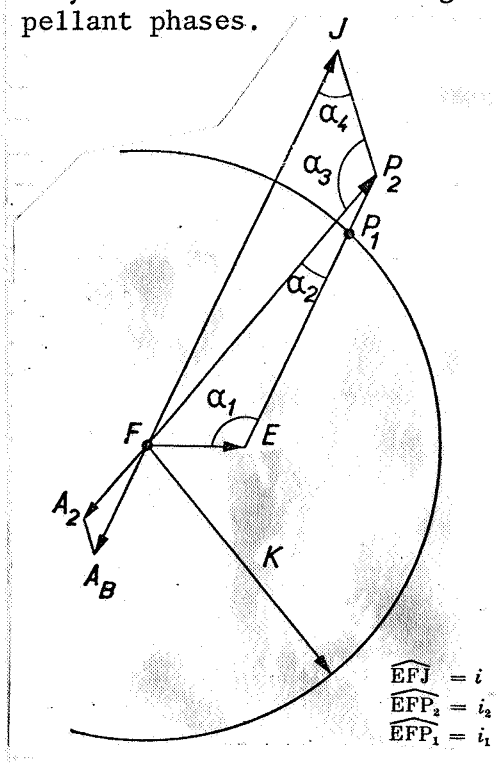

Fig. 11 Hodographic Diagram

The first one leads from the velocity $\vec{V}_0 = \vec{FE}$ (Fig. 11) in the equatorial plane to the perigee velocity of orbit 0_2:

$$\vec{V}_{P_2} = \vec{FP_2}; \left(FP_2 \simeq K\sqrt{\frac{A_B}{(R + A_B)}}\right),$$

the second one leads from the apogee velocity of orbit 0_2:

$$\vec{V}_{A_2} = \vec{FA_2}; \left(FA_2 \simeq K\sqrt{\frac{2R^2}{A_B(R + A_B)}}\right)$$

to the apogee velocity of the target orbit:

$$\vec{V}_{A_B} = \vec{FA_B}; \left(FA_B = K\sqrt{\frac{2P_BR}{A_B(A_B + P_B)}}\right).$$

From the impulse viewpoint, the optimization therefore consists in finding angle i_2 such that

$$C_B = |\overrightarrow{EP_2}| + |\overrightarrow{A_2 A_B}|$$

is minimum. The point P_1 on segment EP_2 and at distance K from F defines in this case the optimum inclinations of the intermediate parking orbit ($i_1 = \widehat{EFP_1}$).

If we grant $\overrightarrow{FJ} = -\frac{A_B}{R}\overrightarrow{FA_B}$ the minimization of $|\overrightarrow{EP_2}| + |\overrightarrow{A_2 A_B}|$ is identical to that of $|\overrightarrow{EP_2}| + \frac{R}{A_B}|\overrightarrow{P_2 J}|$ which is a problem of ordinary optics.

The angles $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are obtained in this way and are those defined on Figure 11.

$$\left.\begin{array}{c} E \sin\alpha_1 = V_{P_2} \sin\alpha_2 = V_{A_2}\sin\alpha_3 = V_{A_B}\sin\alpha_4 \\ \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 360^\circ - i_B \\ \alpha_1 + \alpha_2 = 180^\circ - i_2 \\ \alpha_2 \text{ et } \alpha_4 \text{ sont aigus.} \end{array}\right\} \quad (1)$$

are acute

if:

$$\left|\frac{1}{E} - \frac{1}{V_{A_2}}\right| \geqslant \frac{1}{V_{P_2}} + \frac{1}{V_{A_B}}$$

or even if:

$$i_B \leqslant \text{Arc cos}\,\frac{\min[E, V_{A_2}]}{V_{P_2}} + \text{Arc cos}\,\frac{\min[E, V_{A_2}]}{\max[E, V_{A_B}]},$$

trajectory

which is almost always the case. The equations and inequalities (1) have one root and one alone and the theoretical problem is completed.

Otherwise the equations and inequalities (1) have either one or three roots. In the latter case, the central root corresponds to a relative maximum of C_B, hence having no interest. The two other roots correspond to minimums which are equal if $E = V_{A2}$ (in this case it follows that $\alpha_1 + \alpha_3 = 180°$). Otherwise, the absolute minimum is obtained for $\alpha_3 > 90°$ if $V_{A2} > E$ and $\alpha_1 > 90°$ if $V_{A2} < E$ (each time there is one root and one alone).

$E < V_{A2}$ corresponds to $A_B < R\left[\sqrt{\frac{2K^2}{E^2} + \frac{1}{4}} - \frac{1}{2}\right]$, or $A_B < 23.5$ R in the case of the Earth.

The relative minimum can correspond to the optimum in the case where the launch sector of the launching base is limited.

Investigation for $\varphi \neq 0°$.

Let i_{1M} be the inclination corresponding to the absolute minimum and i_{1m} the one corresponding to the possible relative minimum.

The optimum value of i_1 is:

$$i_1 = i_{1M} \text{ si } \sin i_M \geqslant \sin \varphi.$$
$$i_1 = \varphi \text{ ou } 180^\circ - \varphi \text{ ou } i_{1m}$$

if $\sin i_M < \sin \varphi$, the three cases should be compared (this comparison is never favorable to $180^\circ - \varphi$ if $i_B < 180^\circ \varphi$ and it is never favorable to i_{1m} if $E < V_{A2}$).

In the case in which the launch sector of the launching base is limited, the optimum value of i_1 is i_{1M} (if this value is directly accessible) or, otherwise, i_{1m} or one of the limit values.

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OPTIMIZATION OF THE EXTRA-ATMOSPHERIC PART OF THE ASCENDING TRAJECTORY TO AN ORBIT

PART THREE - THE TARGET ORBIT IS HYPERBOLIC (DEPARTURE FOR AN INTERPLANETARY MISSION)

Christian Marchal

ABSTRACT. When impulses are allowed, optimal ascents are always impulsional only. The optimum solution is almost always of the semi-direct or direct type. Losses due to thrust limitation are moderate for the first impulse and very slight for the other pulses. In connection with the problem the kinematic characteristics of the Hohmann transfers in the solar system are given and the main advantages and disadvantages of intermediate space bases are discussed.

INTRODUCTION

The ascent into orbit is an essential phase of any space mission.

In many cases it is necessary to acquire a velocity greater than the escape velocity. For example, if the target is outside of the "sphere of attraction" of the earth, the geocentric target orbit is then a hyperbola.

The optimization of the ascent into orbit includes two very different parts:

1. Optimization of the crossing of the atmosphere.

2. Optimization of the extra-atmospheric phase of the ascent into orbit.

The first problem, and by far the most complicated of the two, formed the subject of many studies (e.g., reference [1]). It is plainly closely linked with aerodynamics.

The second problem, alone studied here, leads to similar solutions for all high-thrust vehicles (those clearly necessary for an ascent into orbit). The factor optimized is the expenditure of propellant, this leading to the use of solutions whose characteristic velocity is minimum.

The study leads to impulsional solutions which can be carried out in practice, with a slight loss, by means of short-duration thrust trajectories.

Normally, of course, it is required to optimize the aggregate of the two /4 operations at the same time: crossing of the atmosphere, then extra-atmospheric flight path. This generally leads to the use of atmospheric exit velocities which are slightly inclined to the horizontal. This is, however, not always the case (e.g., if only western launching azimuths are available. Also, out of concern for the general aspect, nothing special will be assumed concerning the velocity of "atmospheric exit" considered as initial velocity in the problem examined (we shall nevertheless assume that it is always a matter of an elliptical or parabolic velocity).

III.1 Description of the Theoretical Problem

Upon "exiting from the atmosphere" (or at least from its aerodynamically important layers, or toward 40 to 70 km of altitude in the case of Earth, a moving body is propelled by a determined absolute planetocentric velocity $\vec{V}_o$. How can it be made to reach, with minimum expenditure of propellant, a determined final hyperbolic orbit of asymptotic velocity $\vec{V}_f$. (exit velocity from the "sphere of attraction").

We shall assume that: 1. The attracting planet has spherical symmetry; 2. The velocity $\vec{V}_o$ is elliptical or parabolic; 3. The influence of external heavenly bodies can be disregarded.

In the case of vehicles for which maximum thrust corresponds to the utilization of the maximum ejection velocity (chemical rockets, nuclear rocket, rockets with non-variable ejection velocity, etc...) the minimum expenditure of propellant always corresponds to the use of minimum characteristic velocity solutions.

We shall therefore investigate the optimum ascents from the viewpoint of minimum characteristic velocity, first of all when the thrust is not limited (this leads to impulse solutions) then we shall calculate the loss, generally slight, when the maximum thrust is limited.

We shall disregard the case of rockets for which the maximum thrust does not correspond to the utilization of maximum ejection velocity (nuclear electric rockets with variable ejection velocity, etc...) for these rockets generally supply a low thrust impossible to use for an ascent into orbit.

III.2 Symbols

In order that we might conform to the most widespread usage, we shall reserve the index 0 for initial ("original") conditions, index f for final conditions and indices 1, 2, 3...for thrust phases and for successive intermediate orbits used.

$\vec{V}_0$ = initial absolute planetocentric velocity ("exit velocity from the atmosphere").

Φ_0 = inclination of $\vec{V}_0$ to the horizontal plane (Fig. 1).

$\vec{V}_f$ = final velocity (planetocentric exit velocity from the "sphere of attraction").

C_f = characteristic velocity = arithmetical sum of all the artificial changes of velocity.

φ_f = planetocentric latitude of the direction of $\vec{V}_f$ $(-90^{\circ} \leqslant \varphi_f \leqslant 90^{\circ})$.

R = radius of the atmosphere (= 6370 km + 40 to 70 km in the case of the Earth).

$L = \sqrt{\frac{2\mu}{R}}$ = escape velocity at the level of the "atmospheric exit" = (11.13 km/sec in the case of the Earth).

$K = \sqrt{\frac{\mu}{R}}$ $(= L/\sqrt{2})$ = low circular velocity (= 7.87 km/sec for the earth).

$U_f = \sqrt{L^2 + V_f^2}$ =: grazing velocity for a velocity V_f at infinity.

α_f = trajectory sin $\frac{L^2}{L^2 + 2V_f^2}$ = semi-angle of deviation for a grazing hyperbola (Fig. 1).

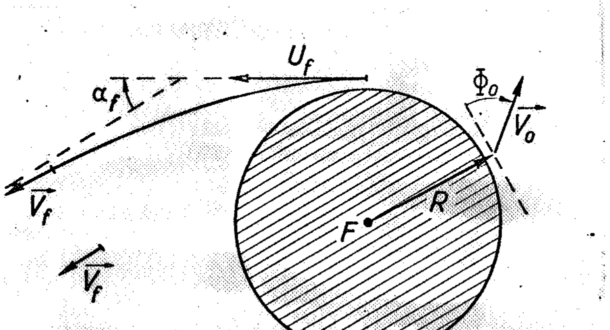

Fig. 1. — Notations :

$U_f = \sqrt{L^2 + V_f^2}$; $\alpha_f = 90^{\circ} - 2$ Arc tg $\frac{V_f}{U_f}$

III.3 Study of the Problem

For once the solution of the problem is simple. Let us consider it indeed from the energy viewpoint alone in the absolute planetocentric axis. It is necessary to go from the energy corresponding to the velocity $\vec{V}_0$ at the "atmospheric exit" to the one corresponding to the velocity $\vec{V}_f$ at the exit from the "sphere of attraction" (or even to the velocity U_f at the level of the "atmospheric exit"). The economical changes of energy are made by the lowest possible tangential thrusts. The characteristic velocity C_f of the ascent is therefore at least $(U_f - V_0)$.

We shall see that owing to suitable maneuvers this characteristic velocity $C_f = U_f - V_0$ can always be very closely obtained.

III.4 Results

From the theoretical viewpoint the optimization of the problem at hand is simple (Fig. 2):

1. A first immediate tangential impulse carries the velocity from V_o to L. The orbit 0_1 is therefore a quasi-parabola.

2. Two very far and very small impulses at I_2 and I_3 allow changing from quasi-parabola.

3. At the perigee (grazing) I_4 of the quasi-parabola 0_3, whose orientation is suitable, the 4th impulse takes place tangentially carrying the velocity from L to U_f.

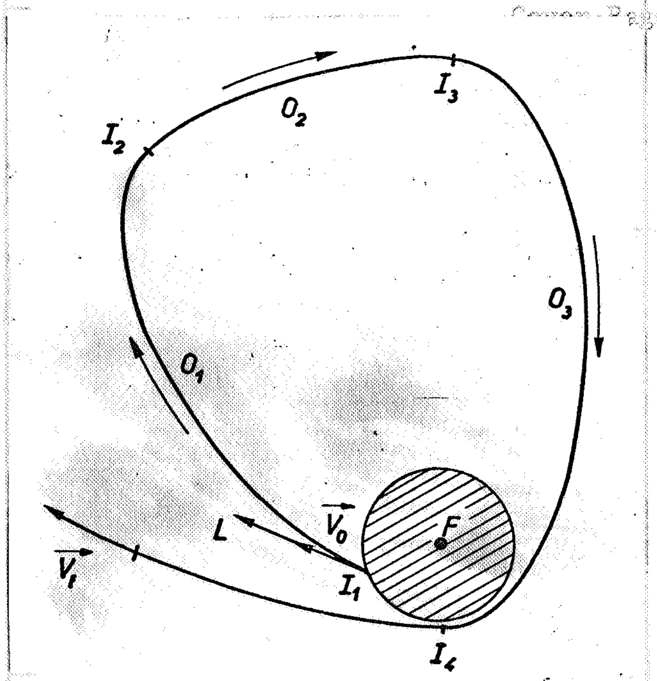

Fig. 2 Theoretical Optimum Ascent

The total characteristic velocity of the ascent is therefore $C_f = U_f - V_0$. /5

In reality, such a solution offers little practical advantage. Indeed, it is necessary to remove I_2 and I_3 to a very great distance in order for the velocity on 0_2 to be low (thus, at the limits of the "sphere of attraction" of the Earth, at approximately r = 240 R, the circular velocity is still on the order of 500 m/s causing all the savings procured by the theoretical solution to be lost).

This is why it is much more advantageous to use the semi-direct solution which commences like the theoretical solution, but at I_2 (located practically at the apogee of 0_1 and 0_2) the small impulse used causes transition to the grazing semi-parabola 0_2 allowing completion of the ascent with one less impulse for the same total theoretical characteristic velocity $C_f = U_f - V_0$.

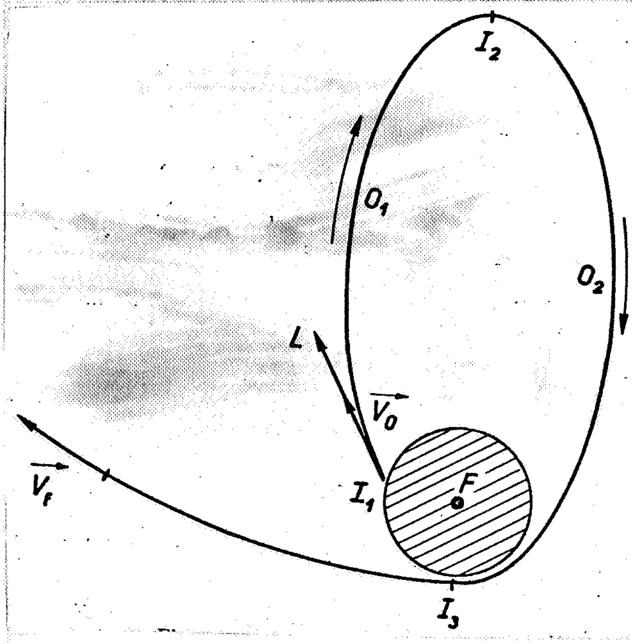

Fig. 3 Semi-Direct Ascent

If φ_2 is the geocentric latitude of I_2, it is possible, by appropriately selecting the launching time and the orientation of the plane of 0_2 around F I_2, to obtain any velocity $\vec{V}_f$ whatever whose direction is included between the planetocentric latitudes $90° - |\varphi_2 - \alpha_f|$. and $|\varphi_2 + \alpha_f| - 90°$.

The chief advantage of the semi-direct solution is that, if D is called the distance from I_2 to the planet center, the impulse required for it is on the order of $L \frac{R}{D}$ instead of $L \sqrt{\frac{R}{D}}$.

Comment I: The loss owing to the non-separation of I_2 at infinity is for all practical purposes $\frac{LR}{D}\sqrt{1 + \cos^2 \Phi_0 - 2 \cos \alpha \cos \Phi_0}$, α being the angle of the planes of 0_1 and 0_2 (planes oriented in the direction of movement). For the Earth in the case D = 60 R the loss is hence on the order of 100 m/s (the duration of roundtrip I_1 I_2 I_3 is in this case 10 days which is completely acceptable given the durations of interplanetary voyages).

Comment II: The optimum altitude of I_3 is a function of the atmosphere. For the Earth it is almost always included between 80 and 120 km. The loss owing to this compromise is on the order of 20 to 80 m/s.

Comment III: If the total impulse $C_f = U_f - V_o$ is applied tangentially at one time to point I_1, the "direct ascent" (Fig. 4) is obtained. It is clearly more practical than the "semi-direct ascent" of Figure 3. However, the direction of $\vec{V}_f$ is imposed in this case (or at least the geocentric latitude of this direction, since the latitude can be selected by means of the launching /6 time). Nevertheless, it is possible to obtain latitudes at 5° or 10° on one hand or the other, for a loss similar to those of the "semi-direct" solution shown above, by using an impulse not completely tangential at I_1.

III.4.2 Special Important Case: $\Phi_0 \simeq 0;\ V_0 < K.$

It is very common for the "atmospheric exit" velocities to be:

1. slightly inclined to the horizontal.

2. less than the low circular velocity K.

It is possible in this case, at the cost of a very slight increase of characteristic velocity, to make transition to an intermediate low parking orbit, thus procuring a great advantage in controllability (the low parking orbits are, moreover, often systematically uses for reasons of technical convenience, precision of later stages, etc...).

Let in this case i_1 be the inclination to the equator and h_1 the mean altitude of parking orbit (whose optimum shape is circular because of losses owing to atmospheric braking). The direct ascent (Fig. 5) with two propelled phases, one immediate for the putting into parking orbit O_1, the other at I_2 for the acquisition of hyperbolic velocity $\vec{V}_f$, has for characteristic velocity:

$$C_f = U_f - V_0 + \frac{Kh_1}{R}\left(1 - \frac{K}{U_f}\right) + \text{ordre } K\,\frac{h_1^2}{R^2}.$$

$$\frac{Kh_1}{R} = 124 \text{ m/s} \cdot \left(\frac{h_1}{100 \text{ km}}\right)$$

in the case of the Earth. (strictly speaking, since K and R have the values defined in the symbols (III, 2), h_1 should be the altitude above the "atmospheric exit".

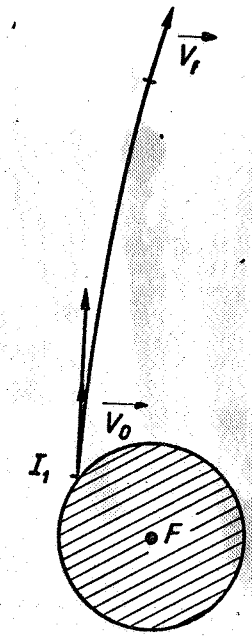

Fig. 4 Direct Ascent

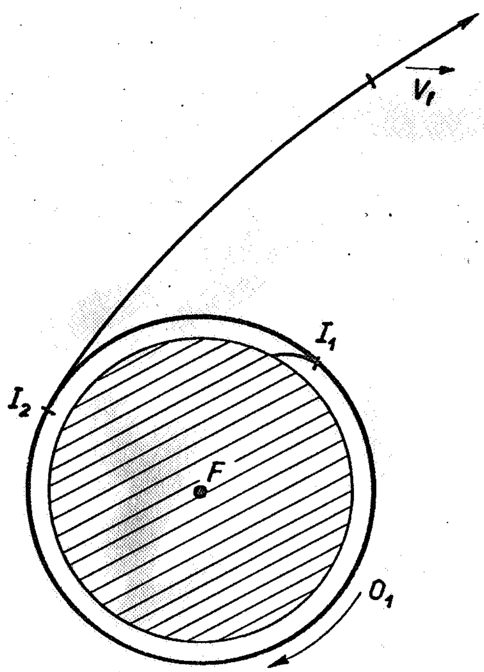

Fig. 5 Direct Ascent-Case: $\Phi_0 \simeq 0;\ V_0 < K.$

This "direct ascent" allows reaching, through selection of launching time and the position of I_2, any velocity $\vec{V}_f$ of direction included between the latitudes $+i_1$ and $-i_1$.

Comment: For parking orbits on O_1 not exceeding several days, the optimum altitude h_1 is, in the case of the Earth, almost always included between 100 and 250 km.

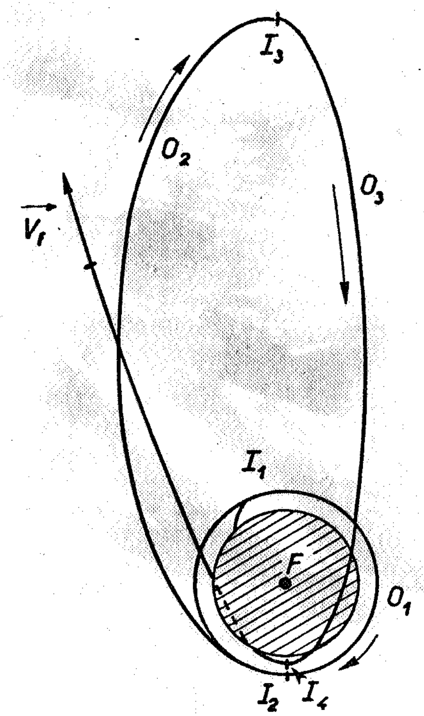

Fig. 6 Semi-Direct Ascent-Case: $\Phi_0 \simeq 0; V_0 < K$.

For higher latitudes the "semi-direct ascent" (Fig. 6) similar to that of Figure 3 costs for practical purposes:

$$C_f = U_f - V_0 + \frac{Kh_1}{R}\left(1 - \frac{\sqrt{2}}{2}\right) + \frac{2\,LR}{D}\sin\frac{\alpha}{2}$$

D being the distance F I_3 and

α the angle of the planes (oriented) of orbits O_2 and O_3 ($0^\circ \leqslant \alpha \leqslant 90^\circ$).

can always be selected).

$\frac{Kh_1}{R}\left(1 - \frac{\sqrt{2}}{2}\right)$ only provides 36 m/s $\left[\frac{h_1}{100\text{ km}}\right]$ in the case of the Earth. This "semi-direct ascent" allows reaching all the φ_f latitudes such as:

$$\alpha_f - i_1 - 90^\circ \leqslant \varphi_f \leqslant 90^\circ + i_1 - \alpha_f.$$

Comment I: As in the case of Figure 3, the optimum altitude of I_4 (above the ground) is included between 80 and 120 km for ordinary vehicles and the terrestrial atmosphere (whence a supplementary loss of 20 to 80 m/s to be added into C_f).

Comment II: For latitudes not much greater than i_1, it is not necessary to use the "semi-direct ascent". The use of a "direct ascent" which is not completely flat (Fig. 7) allows reaching latitudes $\pm (i_1 + \delta i_1)$ for an increase of characteristic velocity of: /7

$$\delta C_f = \frac{K}{2}\,\delta i^2 \left[\frac{(K^2 + V_f^2)\,(U_f + K)}{U_f V_f^2}\right].$$

Or some hundreds of m/s for $\delta i = 5°$ in the ordinary terrestrial cases.

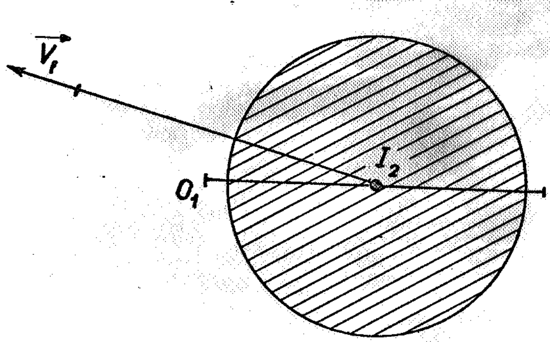

Fig. 7 Direct Ascent-Quasi-Flat Case

III.5 Losses Owing to Limitations of Thrust

These losses have been calculated in [2]. They are rather simple in first approximation (assuming, of course, that the thrust trajectories are optimally arranged).

1. Case of the immediate and non-horizontal thrust (Fig. 2 and 3):

$$\delta C_f \simeq \frac{(L - V_0)^2}{2\,L} \sin \Phi_0 \cdot \frac{g}{\gamma}$$

g designating the acceleration of gravity at the level of "atmospheric exit" and γ the mean acceleration of thrust during the propelled trajectory (γ being assumed only slightly variable).

Comment: This expression of δC_f allows a relative error less than $\frac{g}{15\gamma \sin \Phi_0}$ whereas $V_0 > \frac{L}{10}$ and $\gamma > g$.

In the case of Figure 4, δC_f becomes:

$$\delta C_f \simeq \frac{(U_f - V_0)^2}{2\,U_f} \sin \Phi_0 \cdot \frac{g}{\gamma}$$

2. Case of horizontal or non-immediate thrusts.

The loss is much less in this case. For practical purposes if t is called the duration of the propelled trajectory and T that of the circular revolution at the impulse altitude, the relative loss does not exceed $\frac{\pi^2}{8}\frac{t^2}{T^2}$ for close impulses and $\frac{\pi^2}{6}\frac{t^2}{T^2}$ for far impulses.

III.6 Comparison of Various Launching Bases

Since the characteristic velocity of ascent into orbit is always approximately equal to $(U_f - V_0)$, the comparison of the various bases is reduced to that of the magnitude of the velocities V_0 that can be produced for this purpose for a given rocket. This time again the equatorial bases have the most advantages. The differences, nevertheless are much less than in [3]. They are reduced to those caused by the planet's rotation (or 465 m/s between a polar base and an equatorial terrestrial base). The best "exits" are the ones oriented towards the East and slightly inclined to the horizontal.

III.7 The Interplanetary Missions

The table, on the following page, provides, for purposes of documentation, the chief characteristics of conventional "Hohmann trasfers" (Fig. 8) from Earth toward planets of the solar system. (The planetary orbits are considered as being circular and coplanar. The thickness of the atmospheres is disregarded and the transfer orbits are ellipses bitangential to the planetary orbits).

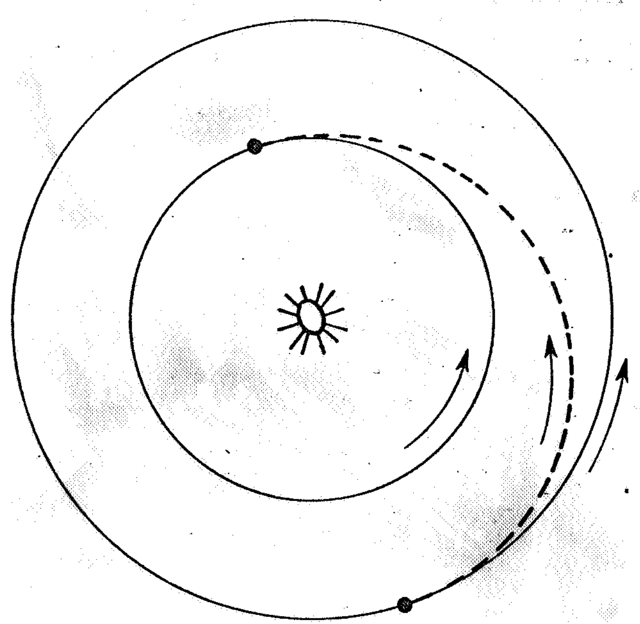

Fig. 8 Hohmann Transfers

Comment I: When atmospheric braking is used upon arrival at the target planet, $U_{f\,Earth} - E_{Earth}$ designates the characteristic velocity of a journey out (equatorial departure) and $U_{f\,Planet} - E_{Planet}$ that for a return trip. The sum $U_{fT} + U_{fP} - E_T - E_P$ designates the characteristic velocity of a round trip (or that of a simple departure without atmospheric braking).

Comment II: The α_f angles are sufficiently small for it to be almost always possible to utilize the "semi-direct ascents" (Figs. 3 and 6) even in the case where transition is made to an equatorial parking orbit ($i_1 = 0$). Indeed, the direction of target exit $\vec{V}_f$ is approximately in the plane of the ecliptic (there is, therefore, in the case of Earth a latitude φ_f included between -24° and + 24°).

Comment III: Being given, for the planets from Mercury to Jupiter, the slight difference in velocities of equatorial and tropical rotation (less than 40 m/s) and the advantages gained through the simplicity of the "direct ascent" (Fig. 5) it is probable that this latter one will be preferred in actual practice (with utilization of inclinations i_1 equal or slightly grater than $|\varphi_f|$, unless the danger presented by radiation belts obliges the selection of large values of i_1).

The Velocities shown are in km/sec to Approximately 0.1

		Mercury	Venus	Earth	Mars	Small Planets		Jupiter	Saturn	Uranus	Neptune	Pluto
The Distance to the Sun [Earth = 1]		0,387	0,723	1	1,524	2	3	5,20	9,55	19,22	30,11	39,44
Equatorial Rotation Velocity = E		0	0?	0,5	0,25	0	0	12,6	10,2	4,1	2,5	?
Satellizing Velocity = K		3,1	7,3	7,9	3,5	0	0	42,3	25,2	15,1	17,6	3,7 ?
Escape Velocity = L		4,4	10,3	11,2	5,0	0	0	59,8	35,6	21,4	24,9	5,2 ?
Planet's Orbital Velocity		47,8	35,0	29,77	24,1	21	17,2	13,0	9,6	6,8	5,4	4,7
Hohmann Transfer: Earth → planet or planet → Earth	V_f. Earth	7,5	2,5		3,0	4,6	6,7	8,8	10,3	11,3	11,6	11,8
	U_f. Earth	13,5	11,5		11,6	12,1	13,1	14,2	15,2	15,9	16,1	16,3
	α_f. Earth	32°	65°		61°	48°	36°	26°	22°	19°	18°	18°
	V_f. Planet	9,6	2,7		2,6	3,9	5,1	5,6	5,4	4,7	4	3,6
	U_f. Planet	10,5	10,7		5,6	3,9	5,1	60,0	36,0	21,9	25,2	6,3 ?
	α_f. Planet	5°	62°		40°	0°	0°	79°	73°	66°	72°	30?
Inclination of the Equator to the Orbiting Plane		?	?	23°27′	25°	—	—	3°	27°	98°	30°	?
U_fEarth $-E_{Earth}$		13,0	11,0		11,1	11,6	12,6	13,7	14,7	15,4	15,6	15,8
U_fPlanet $-E_{Planet}$		10,5	10,7?		5,4	3,9	5,1	47,4	25,8	17,8	22,7	6,3?
$U_{fT} + U_{fP} - E_T - E_P$		23,5	21,7?		16,5	15,5	17,7	61,1	40,5	33,2	38,3	22,1?
Duration of Hohmann Transfer (in years)		0,290	0,400		0,71	0,92	1,42	2,70	6,1	16	31	45,5

III.8 Utilization of Bases in Orbit /8

The utilization of large intermediate bases in orbit around the planets offers a specific economic advantage by allowing specialization of transportation rockets: some for planet-base connections, having a specially designed aerofoil; others for interplanetary connections from base to base, equipped for voyages of long duration and only passing through the atmosphere for the sake of atmospheric braking during arrival in the proximity of a planet (thus allowing transition to be made practically without cost from the descent hyperbola to the orbit of the base if the latter is near the atmosphere and if its plane is near the direction of arrival).

Take note that, from the viewpoint of characteristic velocity, the rockets performing the short planet-base journey will require, by far, the greatest /9 expenditure (hence the great advantage of research involving recoverable stages, hypersonic glider, etc...). On the other hand, departing from a non-secant orbit with the attracting planet, the rockets carrying out the interplanetary part of the voyage will be able to conveniently use the low-thrust motors (nuclear-electric, etc...) if the latter are found to be more economical.

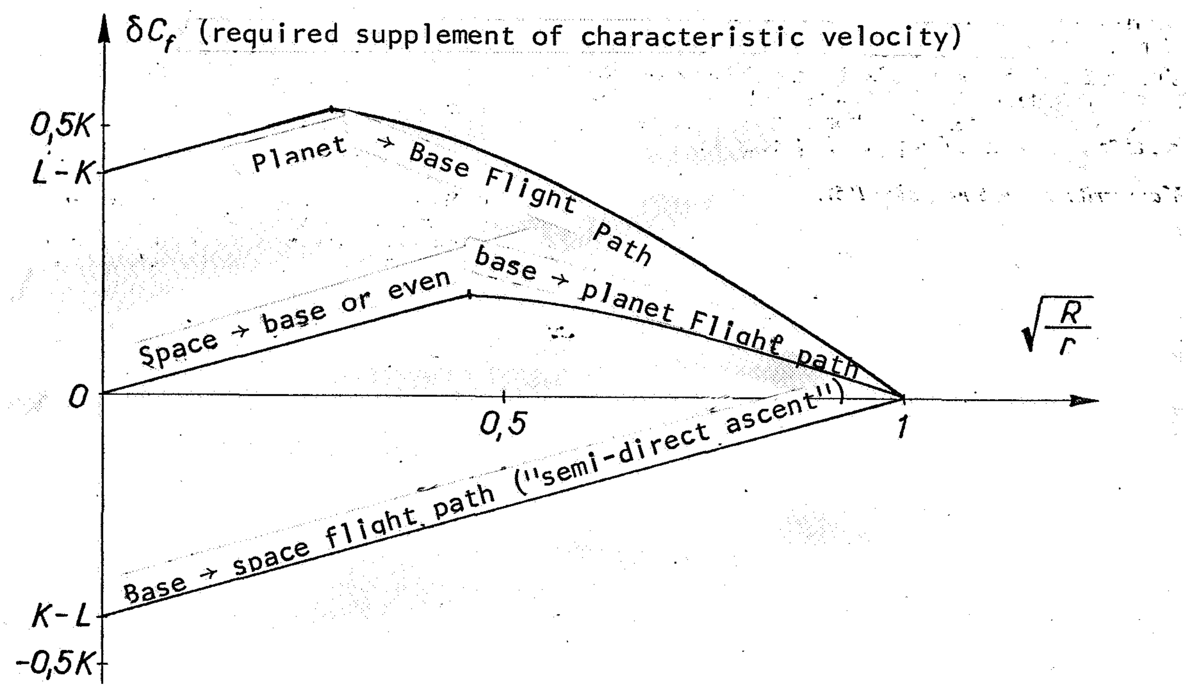

Fig. 9 Losses owing to Utilization of an Intermediate Space Base with Circular Orbit Whose Radius is r.

The optimum position of base orbit is as low as possible taking into account the atmosphere; for the Earth an orbit at approximately 600 km of altitude, enduring a service life on the order of 50 years and located sufficiently below the radiation belts, seems quite suitable.

There has been plotted on Figure 9, as a function of radius r of the base orbit (disregarding the thickness of the atmosphere), those losses of characteristic velocity owing to the establishment of this base on a far orbit.

Comment I: The descending flight paths (space → base and base-planet), flight paths both providing the same loss, are assumed to utilize atmospheric brakings. This allows, furthermore, the convenient carrying-out of important changes of orbiting plane at no cost, even for low aerodynamic lift-drag ratios.

Comment II: The base → space flight path shows not a loss but a gain. It is, nevertheless, insufficient to compensate for the other losses (except for when r quite large, a case with few advantages owing to delays it requires).

Comment III: The planet → base and base → planet flight paths have been optimized in the manner shown in [2].

Comment IV: The case of a base → space ascent of the "direct ascent" type (Figs. 5 and 7) and not the "semi-direct ascent" (Fig. 6) was not shown because it is a function of the value and the direction of $\vec{V}_f$ and is never more favorable than the "semi-direct ascent"(from the viewpoint of characteristic velocity).

III.9 Utilization of Intermediate Bases on Natural Satellites

The utilization of intermediate bases on natural satellites offers important physical advantages: suitable protection, in subterranean shelters, agains meteorites and radiations, location on the spot of some raw materials, etc... These advantages should be compared to the losses which can be read on Figure 9 for plainly r is required in this case.

There is thus found a loss of approximately 1 km/sec for satellites of Mars, in the Mars-Space direction as well as in the Space-Mars direction (in addition, since these satellites are close to the plane of the Martian equator, therefore at approximately 25° from the plane of the Mars orbit, maneuvers of the "semi-direct ascent" type are required in the Mars-Space direction).

As far as our heavy satellite the Moon is concerned, the loss is much higher. It is close to 5 km/sec in a direction as in the other (the lunar escape velocity is 2.4 km/sec). Thus, it is probable that lunar intermediate bases will only be utilized whenever the Moon forms the subject of a considerable colonization.

CONCLUSION

The study of the optimization of the extra-atmospheric phase of the ascent into orbit when the target orbit is hyperbolic (departure for an interplanetary mission) is a relatively simple problem leading to solutions including at the most four propelled phases.

The "direct exit" with two propelled phases and a low intermediate parking orbit generally forms the most practical economic solution.

The cost of ascents into orbit is almost independent of the orientation. /10

The utilization of intermediate space bases in orbit around planets is very advantageous, especially if these bases are close. On the other hand, the utilization of bases on natural satellites leads to considerable losses.

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